THE ORBIT STRUCTURE OF THE GELFAND-ZEITLIN GROUP ON $n \times n$ MATRICES

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Dedicated to Bertram Kostant on the occasion of his 80th birthday.

Abstract. In recent work ([9],[10]), Kostant and Wallach construct an action of a simply connected Lie group $A \simeq \mathbb{C}^{\binom{n}{2}}$ on $\mathfrak{gl}(n)$ using a completely integrable system derived from the Poisson analogue of the Gelfand-Zeitlin subalgebra of the enveloping algebra. In [9], the authors show that A-orbits of dimension $\binom{n}{2}$ form Lagrangian submanifolds of regular adjoint orbits in $\mathfrak{gl}(n)$. They describe the orbit structure of A on a certain Zariski open subset of regular semisimple elements. In this paper, we describe all A-orbits of dimension $\binom{n}{2}$ and thus all polarizations of regular adjoint orbits obtained using Gelfand-Zeitlin theory.

1. Introduction

In recent papers ([9], [10]), Bertram Kostant and Nolan Wallach construct an action of a complex, commutative, simply connected Lie group $A \simeq \mathbb{C}^{\binom{n}{2}}$ on the Lie algebra of $n \times n$ complex matrices $\mathfrak{gl}(n)$. The dimension of this group is exactly half the dimension of a regular adjoint orbit in $\mathfrak{gl}(n)$ and orbits of A of dimension $\binom{n}{2}$ are Lagrangian submanifolds of regular adjoint orbits. We refer to the group A introduced by Kostant and Wallach as the Gelfand-Zeitlin group, because of its connection with the Gelfand-Zeitlin algebra, as we will explain in section 2.

The group A and its action are constructed as follows. Given $i < n$, we can think of $\mathfrak{gl}(i) \hookrightarrow \mathfrak{gl}(n)$ as a subalgebra by embedding an $i \times i$ matrix into the top left-hand corner of an $n \times n$ matrix. For $1 \leq i \leq n$ and $1 \leq j \leq i$, let $f_{i,j}(x)$ be the polynomial on $\mathfrak{gl}(n)$ defined by $f_{i,j}(x) = tr(x_i^j)$, where x_i denotes the $i \times i$ submatrix in the top left-hand corner of x. In [9], it is shown that the functions $\{f_{i,j} | 1 \leq i \leq n, 1 \leq j \leq i\}$ are algebraically independent and Poisson commute with respect to the Lie-Poisson structure on $\mathfrak{gl}(n) \simeq \mathfrak{gl}(n)^*$. The corresponding Hamiltonian vector fields $\xi_{f_{i,j}}$ generate a commutative Lie algebra $\mathfrak{a}$ of dimension $\binom{n}{2}$. The group A is defined to be the simply connected, complex Lie group that corresponds to the Lie algebra $\mathfrak{a}$. The vector fields $\xi_{f_{i,j}}$ are complete (Theorem 3.5 in [9]), and therefore $\mathfrak{a}$ integrates to a global action of $\mathbb{C}^{\binom{n}{2}}$ on $\mathfrak{gl}(n)$. This action of $\mathbb{C}^{\binom{n}{2}}$ defines the action of the group A on $\mathfrak{gl}(n)$.

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Our goal in this paper is to describe all A-orbits of dimension $\binom{n}{2}$. An element $x \in \mathfrak{gl}(n)$ is called strongly regular if and only if its A-orbit is of dimension $\binom{n}{2}$. One way of studying such orbits is to study the action of A on fibres the map $\Phi : \mathfrak{gl}(n) \to \mathbb{C}^{\frac{n(n+1)}{2}}$

$$\Phi(x) = (p_{1,1}(x_1), p_{2,1}(x_2), \cdots, p_{n,n}(x)), \tag{1.1}$$

where $p_{i,j}(x_i)$ is the coefficient of t^{j-1} in the characteristic polynomial of x_i.

In Theorem 2.3 in [9], the authors show that this map is surjective and that every fibre of this map $\Phi^{-1}(c) = \mathfrak{gl}(n)_c$ contains strongly regular elements. Following [9], we denote the strongly regular elements in the fibre $\mathfrak{gl}(n)_c$ by $\mathfrak{gl}(n)_c^{sreg}$. By Theorem 3.12 in [9], the A-orbits in $\mathfrak{gl}(n)^{sreg}$ are precisely the irreducible components of the fibres $\mathfrak{gl}(n)_c^{sreg}$. Thus, our study of the action of A on $\mathfrak{gl}(n)^{sreg}$ is reduced to studying the A-orbit structure of the fibres $\mathfrak{gl}(n)_c^{sreg}$. In [9], Kostant and Wallach describe the A-orbit structure on a special class of fibres that consist of certain regular semisimple elements. In this paper, we describe the A-orbit structure of $\mathfrak{gl}(n)_c^{sreg}$ for any $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$.

In section 2, we describe the construction of the group A in [9] in more detail. In section 3, we describe the results in [9] about its orbit structure. We summarize theses results briefly here. For any $x \in \mathfrak{gl}(i)$, let $\sigma(x)$ denote the spectrum of x. In [9], Kostant and Wallach describe the action of the group A on a Zariski open subset of regular semisimple elements defined by

$$\mathfrak{gl}(n)_\Omega = \{x \in \mathfrak{gl}(n) |\ x_i \text{ is regular semisimple, } \sigma(x_{i-1}) \cap \sigma(x_i) = \emptyset,\ 2 \leq i \leq n\}.$$

Let $c_i \in \mathbb{C}^i$ and consider $c = (c_1, c_2, \cdots, c_n) \in \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n = \mathbb{C}^{\frac{n(n+1)}{2}}$. Regard $c_i = (z_1, \cdots, z_i)$ as the coefficients of the degree i monic polynomial

$$p_{c_i}(t) = z_1 + z_2 t + \cdots + z_i t^{i-1} + t^i. \tag{1.2}$$

Let Ω_n denote the Zariski open subset of $\mathbb{C}^{\frac{n(n+1)}{2}}$ given by the tuples c such that $p_{c_i}(t)$ has distinct roots and $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have no roots in common. Clearly, $\mathfrak{gl}(n)_\Omega = \bigcup_{c \in \Omega_n} \mathfrak{gl}(n)_c$. The action of A on $\mathfrak{gl}(n)_\Omega$ is described in the following theorem. (Theorem 3.2).

Theorem 1.1. *The elements of $\mathfrak{gl}(n)_\Omega$ are strongly regular. If $c \in \Omega_n$ then $\mathfrak{gl}(n)_c = \mathfrak{gl}(n)_c^{sreg}$ is precisely one orbit under the action of the group A. Moreover, $\mathfrak{gl}(n)_c$ is a homogeneous space for a free, algebraic action of the torus $(\mathbb{C}^\times)^{\binom{n}{2}}$.*

In section 4, we give a construction that describes an A-orbit in an arbitrary fibre $\mathfrak{gl}(n)_c^{sreg}$ as the image of a certain morphism of a commutative, connected algebraic group into $\mathfrak{gl}(n)_c^{sreg}$. The construction in section 4 gives a bijection between A-orbits in $\mathfrak{gl}(n)_c^{sreg}$ and orbits of a product of connected, commutative algebraic groups acting freely on a fairly simple variety, but it does not enumerate the A-orbits in $\mathfrak{gl}(n)_c^{sreg}$. In section 5, we use the construction developed in section 4 and combinatorial data of the fibre $\mathfrak{gl}(n)_c^{sreg}$ to give explicit descriptions of the A-orbits in $\mathfrak{gl}(n)_c^{sreg}$. The main result is Theorem 5.11, which contrasts substantially with the generic case described in Theorem 1.1.

Theorem 1.2. *Let $c = (c_1, c_2, \cdots, c_n) \in \mathbb{C}^1 \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n = \mathbb{C}^{\frac{n(n+1)}{2}}$ be such that there are $0 \leq j_i \leq i$ roots in common between the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. Then the number of A-orbits in $\mathfrak{gl}(n)_c^{sreg}$ is exactly $2^{\sum_{i=1}^{n-1} j_i}$. For $x \in \mathfrak{gl}(n)_c^{sreg}$, let Z_i denote the centralizer of the Jordan form of x_i in $\mathfrak{gl}(i)$. The orbits of A on $\mathfrak{gl}(n)_c^{sreg}$ are the orbits of a free algebraic action of the complex, commutative, connected algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$ on $\mathfrak{gl}(n)_c^{sreg}$.*

Remark 1.3. After the results of this paper were established, a very interesting paper by Roger Bielawski and Victor Pidstrygach appeared in [1] proving similar results. The arguments are completely different, and the proofs were formed independently. In [1], the authors define an action of A on the space of rational maps of fixed degree from the Riemann sphere into the flag manifold for $GL(n+1)$ and use symplectic reduction to obtain results about the strongly regular set. They also show that there are $2^{\sum_{i=1}^{n-1} j_i}$ A-orbits in $\mathfrak{gl}(n)_c^{sreg}$, c as in Theorem 1.2. Our work differs from that of [1] in that we explicitly list the A-orbits in $\mathfrak{gl}(n)_c^{sreg}$ and obtain an algebraic action of $Z_1 \times \cdots \times Z_{n-1}$ on $\mathfrak{gl}(n)_c^{sreg}$ whose orbits are the same as those of A. In spite of the relation between these papers, we feel that our paper provides a different and more precise perspective on the problem and deserves a place in the literature.

The nilfibre $\mathfrak{gl}(n)_0 = \Phi^{-1}(0)$ contains some of the most interesting structure in regards to the action of A. The fibre $\mathfrak{gl}(n)_0$ has been studied extensively by Lie theorists and numerical linear algebraists. Parlett and Strang [12] have studied matrices in $\mathfrak{gl}(n)_0$ and have obtained interesting results. Ovsienko [11] has also studied $\mathfrak{gl}(n)_0$, and has shown that it is a complete intersection. It turns out that the A-orbits in $\mathfrak{gl}(n)_0^{sreg}$ correspond to 2^{n-1} Borel subalgebras of $\mathfrak{gl}(n)$. The main results are contained in Theorems 5.2 and 5.5. We combine them into one single statement here.

Theorem 1.4. *The nilfibre $\mathfrak{gl}(n)_0^{sreg}$ contains 2^{n-1} A-orbits. For $x \in \mathfrak{gl}(n)_0^{sreg}$, let $\overline{A \cdot x}$ denote the Zariski (=Hausdorff) closure of $A \cdot x$. Then $\overline{A \cdot x}$ is a nilradical of a Borel subalgebra in $\mathfrak{gl}(n)$ that contains the standard Cartan subalgebra of diagonal matrices.*

The nilradicals obtained as closures of A-orbits in $\mathfrak{gl}(n)_0^{sreg}$ are described explicitly in Theorem 5.5. We also describe the permutations that conjugate the strictly lower triangular matrices into each of these 2^{n-1} nilradicals in Theorem 5.7.

Theorem 1.2 lets us identify exactly where the action of the group A is transitive on $\mathfrak{gl}(n)_c^{sreg}$. (See Corollary 5.13 and Remark 5.14).

Corollary 1.5. The action of A is transitive on $\mathfrak{gl}(n)_c^{sreg}$ if and only if $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ are relatively prime for each i, $1 \leq i \leq n-1$. Moreover, for such $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$ we have $\mathfrak{gl}(n)_c = \mathfrak{gl}(n)_c^{sreg}$.

This corollary allows us to identify the maximal subset of $\mathfrak{gl}(n)$ on which the action of A is transitive on the fibres of the map Φ in (1.1) over this set. The set $\mathfrak{gl}(n)_\Omega$ is a proper open subset of this maximal set. This is discussed in detail in section 5.3.

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2. The Group A

We briefly discuss the construction of an analytic action of a group $A \simeq \mathbb{C}^{\binom{n}{2}}$ on $\mathfrak{gl}(n)$ that appears in [9] (see also [3]).

We view $\mathfrak{gl}(n)^*$ as a Poisson manifold with the Lie-Poisson structure (see [14], [2]). Recall that the Lie Poisson structure is the unique Poisson structure on the symmetric algebra $S(\mathfrak{gl}(n)) = \mathbb{C}[\mathfrak{gl}(n)^*]$ such that if $x, y \in S^1(\mathfrak{gl}(n))$, then their Poisson bracket $\{x, y\} = [x, y]$ is their Lie bracket. We use the trace form to transfer the Poisson structure from $\mathfrak{gl}(n)^*$ to $\mathfrak{gl}(n)$. For $i \leq n$, we can view $\mathfrak{gl}(i) \hookrightarrow \mathfrak{gl}(n)$ as a subalgebra, simply by embedding an $i \times i$ matrix in the top left-hand corner of an $n \times n$ matrix.

$$Y \hookrightarrow \left[\begin{array}{cc} Y & 0 \\ 0 & 0 \end{array}\right]. \tag{2.1}$$

We also have a corresponding embedding of the adjoint groups $GL(i) \hookrightarrow GL(n)$

$$g \hookrightarrow \left[\begin{array}{cc} g & 0 \\ 0 & Id_{n-i} \end{array}\right].$$

For the purposes of this paper, we always think of $\mathfrak{gl}(i) \hookrightarrow \mathfrak{gl}(n)$ and $GL(i) \hookrightarrow GL(n)$ via these embeddings, unless otherwise stated.

We can use the embedding (2.1) to realize $\mathfrak{gl}(i)$ as a summand of $\mathfrak{gl}(n)$. Indeed, we have

$$\mathfrak{gl}(n) = \mathfrak{gl}(i) \oplus \mathfrak{gl}(i)^{\perp}, \tag{2.2}$$

where $\mathfrak{gl}(i)^{\perp}$ denotes the orthogonal complement of $\mathfrak{gl}(i)$ in $\mathfrak{gl}(n)$ with respect to the trace form. It is convenient for us to have a coordinate description of this decomposition. We make the following definition.

Definition 2.1. For $x \in \mathfrak{gl}(n)$, we let $x_i \in \mathfrak{gl}(i)$ be the top left-hand corner of x, i.e. $(x_i)_{k,l} = x_{k,l}$ for $1 \leq k, l \leq i$. We refer to x_i as the $i \times i$ cutoff of x.

Given a $y \in \mathfrak{gl}(n)$ its decomposition in (2.2) is written $y = y_i \oplus y_i^{\perp}$ where $y_i^{\perp}$ denotes the entries $y_{k,l}$ where k, l are not both in the set $\{1, \cdots, i\}$. Using the decomposition in (2.2), we can think of the polynomials on $\mathfrak{gl}(i)$, $P(\mathfrak{gl}(i))$, as a Poisson subalgebra of the polynomials on $\mathfrak{gl}(n)$, $P(\mathfrak{gl}(n))$. Explicitly, if $f \in P(\mathfrak{gl}(i))$, (2.2) gives $f(x) = f(x_i)$ for

$x \in \mathfrak{gl}(n)$. The Poisson structure on $P(\mathfrak{gl}(i))$ inherited from $P(\mathfrak{gl}(n))$ agrees with the Lie-Poisson structure on $P(\mathfrak{gl}(i))$ (see [9], pg. 330).

Since $\mathfrak{gl}(n)$ is a Poisson manifold, we have the notion of a Hamiltonian vector field ξ_f for any holomorphic function $f \in \mathcal{O}(\mathfrak{gl}(n))$. If $g \in \mathcal{O}(\mathfrak{gl}(n))$, then $\xi_f(g) = \{f, g\}$. The group A is defined as the simply connected, complex Lie group that corresponds to a certain Lie algebra of Hamiltonian vector fields on $\mathfrak{gl}(n)$. To define this Lie algebra of vector fields, we consider the subalgebra of $P(\mathfrak{gl}(n))$ generated by the adjoint invariant polynomials for each of the subalgebras $\mathfrak{gl}(i)$, $1 \leq i \leq n$.

$$J(\mathfrak{gl}(n)) = P(\mathfrak{gl}(1))^{GL(1)} \otimes \cdots \otimes P(\mathfrak{gl}(n))^{GL(n)}. \tag{2.3}$$

This algebra may be viewed as a classical analogue of the Gelfand-Zeitlin subalgebra of the universal enveloping algebra $U(\mathfrak{gl}(n))$ (see [5]). As $P(\mathfrak{gl}(i))^{GL(i)}$ is in the Poisson centre of $P(\mathfrak{gl}(i))$, it is easy to see that $J(\mathfrak{gl}(n))$ is Poisson commutative. (See Proposition 2.1 in [9].) Let $f_{i,1}, \cdots, f_{i,i}$ generate the ring $P(\mathfrak{gl}(i))^{GL(i)}$. Then $J(\mathfrak{gl}(n))$ is generated by $\{f_{i,1}, \cdots, f_{i,i} | 1 \leq i \leq n\}$. Note that the sum

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = \binom{n}{2}$$

is half the dimension of a regular adjoint orbit in $\mathfrak{gl}(n)$. We will see shortly that the functions $\{f_{i,1}, \cdots, f_{i,i} | 1 \leq i \leq n-1\}$ form a completely integrable system on a regular adjoint orbit.

The surprising fact about this integrable system proven by Kostant and Wallach in [9] is that the corresponding Hamiltonian vector fields $\{\xi_{f_{i,j}} |\ 1 \leq j \leq i,\ 1 \leq i \leq n-1\}$ are complete (see Theorem 3.5 in [9]). Let $f_{i,j} = tr(x_i^j)$ and let $\mathfrak{a} = \{\xi_{f_{i,j}} |\ 1 \leq j \leq i,$ $1 \leq i \leq n-1\}$. We define A as the simply connected, complex Lie group corresponding to the Lie algebra $\mathfrak{a}$. Since the vector fields $\xi_{f_{i,j}}$ commute for all i and j, the corresponding (global) flows define a global action of $\mathbb{C}^{\binom{n}{2}}$ on $\mathfrak{gl}(n)$. $A \simeq \mathbb{C}^{\binom{n}{2}}$, and it acts on $\mathfrak{gl}(n)$ by composing these flows in any order. The action of A also preserves adjoint orbits. (See [9], Theorems 3.3, 3.4.)

The action of $A \simeq \mathbb{C}^{\binom{n}{2}}$ may seem at first glance to be non-canonical as choices are involved in its definition. However, one can show that the orbit structure of $\mathbb{C}^{\binom{n}{2}}$ given by integrating the complete vector fields $\xi_{f_{i,j}}$ is independent of the choice of generators $f_{i,j}$ for $P(\mathfrak{gl}(i))^{GL(i)}$. (See Theorem 3.5 in [9].) Since we are interested in studying the geometry of these orbits, we lose no information by fixing a choice of generators.

Remark 2.2. Using the Gelfand-Zeitlin algebra for complex orthogonal Lie algebras $\mathfrak{so}(n)$, we can define an analogous group, $\mathbb{C}^d$ where d is half the dimension of a regular adjoint orbit in $\mathfrak{so}(n)$. The construction of the group and the study of its orbit structure on certain regular semisimple elements of $\mathfrak{so}(n)$ is discussed in detail in [3].

For our choice of generators, we can write down the Hamiltonian vector fields $\xi_{f_{i,j}}$ in coordinates and their corresponding global flows. To do this, we use the following

notation. Given x, $z \in \mathfrak{gl}(n)$, we denote the directional derivative in the direction of z evaluated at x by ∂_x^z. Its action on function on a holomorphic function f is

$$\partial_x^z f = \frac{d}{dt}|_{t=0} f(x+tz). \tag{2.4}$$

By Theorem 2.12 in [9]

$$(\xi_{f_{i,j}})_x = \partial_x^{[-jx_i^{j-1},\, x]}. \tag{2.5}$$

We see that $\xi_{f_{i,j}}$ integrates to an action of $\mathbb{C}$ on $\mathfrak{gl}(n)$ given by

$$\mathrm{Ad}(\exp(tjx_i^{j-1}))\cdot x \tag{2.6}$$

for $t \in \mathbb{C}$, where $x_i^0 = Id_i \in \mathfrak{gl}(i)$.

Remark 2.3. The orbits of A are the composition of the (commuting) flows in (2.6) for $1 \le i \le n-1$, $1 \le j \le i$ in any order acting on $x \in \mathfrak{gl}(n)$. It is easy to see using (2.6) that the action of A stabilizes adjoint orbits.

Equation (2.5) gives us a convenient description of the tangent space to the action of A on $\mathfrak{gl}(n)$. We first need some notation. If $x \in \mathfrak{gl}(n)$, let Z_{x_i} be the associative subalgebra of $\mathfrak{gl}(i)$ generated by the elements $Id_i, x_i, x_i^2, \cdots, x_i^{i-1}$. We then let $Z_x = \sum_{i=1}^n Z_{x_i}$. Let $x \in \mathfrak{gl}(n)$ and let $A\cdot x$ denote its A-orbit. Then equation (2.5) gives us

$$T_x(A\cdot x) = span\{(\xi_{f_{i,j}})_x | 1 \le i \le n-1,\ 1 \le j \le i\} = span\{\partial_x^{[z,x]} | z \in Z_x\}.$$

Following the notation in [9], we denote

$$V_x := span\{\partial_x^{[z,x]} | z \in Z_x\} = T_x(A\cdot x) \subset T_x(\mathfrak{gl}(n)). \tag{2.7}$$

Our work focuses on orbits of A of maximal dimension $\binom{n}{2}$, as such orbits form Lagrangian submanifolds of regular adjoint orbits. (If such orbits exist, they are the leaves of a maximal dimension of the Gelfand-Zeitlin integrable system.) Accordingly, we make the following theorem-definition. (See Theorem 2.7 and Remark 2.8 in [9]).

Theorem-Definition 2.4. $x \in \mathfrak{gl}(n)$ is called strongly regular if and only if the differentials $\{(df_{i,j})_x | 1 \le i \le n, 1 \le i \le i\}$ are linearly independent at x. Equivalently, x is strongly regular if the A-orbit of x, $A\cdot x$ has $\dim(A\cdot x) = \binom{n}{2}$. We denote the set of strongly regular elements of $\mathfrak{gl}(n)$ by $\mathfrak{gl}(n)^{sreg}$.

The goal of the paper is to determine the A-orbit structure of $\mathfrak{gl}(n)^{sreg}$. In [9], Kostant and Wallach produce strongly regular elements using the map $\Phi : \mathfrak{gl}(n) \to \mathbb{C}^{\frac{n(n+1)}{2}}$,

$$\Phi(x) = (p_{1,1}(x_1), p_{2,1}(x_2), \cdots, p_{n,n}(x)), \tag{2.8}$$

where $p_{i,j}(x_i)$ is the coefficient of t^{j-1} in the characteristic polynomial of x_i.

One of the major results in [9] is the following theorem concerning Φ. (See Theorem 2.3 in [9].)

Theorem 2.5. *Let $\mathfrak{b} \subset \mathfrak{gl}(n)$ denote the standard Borel subalgebra of upper triangular matrices in $\mathfrak{gl}(n)$. Let f be the sum of the negative simple root vectors. Then the restriction of Φ to the affine variety $f + \mathfrak{b}$ is an algebraic isomorphism.*

We will refer to the elements of $f + \mathfrak{b}$ as Hessenberg matrices. They are matrices of the form

$$f + \mathfrak{b} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ 0 & 1 & \cdots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_{nn} \end{bmatrix}.$$

Note that Theorem 2.5 implies that if $x \in f + \mathfrak{b}$, then the differentials $\{(dp_{i,j})_x | 1 \leq i \leq n,\, 1 \leq j \leq i\}$ are linearly independent. The sets of functions $\{f_{i,j} | 1 \leq i \leq n,\, 1 \leq j \leq i\}$ and $\{p_{i,j} | 1 \leq i \leq n,\, 1 \leq j \leq i\}$ both generate the classical analogue of the Gelfand-Zeitlin algebra $J(\mathfrak{gl}(n))$ (see (2.3)). It follows that for any $x \in \mathfrak{gl}(n)$, $span\{(df_{i,j})_x | 1 \leq i \leq n,\, 1 \leq j \leq i\} = span\{(dp_{i,j})_x | 1 \leq i \leq n,\, 1 \leq j \leq i\}$ by the Leibniz rule. Theorem 2.5 then implies

$$f + \mathfrak{b} \subset \mathfrak{gl}(n)^{sreg} \text{ and therefore } \mathfrak{gl}(n)^{sreg} \text{ is a non-empty Zariski open subset of } \mathfrak{gl}(n).$$

Thus, the functions

$$\{f_{i,j} | 1 \leq i \leq n,\, 1 \leq j \leq i\} \text{ are algebraically independent.}$$

For $c = (c_1, c_2, \cdots, c_n) \in \mathbb{C} \times \mathbb{C}^2 \times \cdots \times \mathbb{C}^n = \mathbb{C}^{\frac{n(n+1)}{2}}$ we denote the fibre $\Phi^{-1}(c) = \mathfrak{gl}(n)_c$, Φ as in (2.8). For $c_i \in \mathbb{C}^i$, we define a monic polynomial $p_{c_i}(t)$ with coefficients given by c_i as in (1.2). $x \in \mathfrak{gl}(n)_c$ if and only if x_i has characteristic polynomial $p_{c_i}(t)$ for all i . Theorem 2.5 says that for any $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$, $\mathfrak{gl}(n)_c$ is non-empty and contains a unique Hessenberg matrix. We denote the strongly regular elements of the fibre $\mathfrak{gl}(n)_c$, by $\mathfrak{gl}(n)_c^{sreg}$ that is

$$\mathfrak{gl}(n)_c^{sreg} = \mathfrak{gl}(n)_c \cap \mathfrak{gl}(n)^{sreg}.$$

Since Hessenberg matrices are strongly regular, we get

$$\mathfrak{gl}(n)_c^{sreg} \text{ is a non-empty Zariski open subset of } \mathfrak{gl}(n)_c$$

for any $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$.

Theorem 2.5 implies that every regular adjoint orbit contains strongly regular elements. This follows from the fact that a regular adjoint orbit contains a companion matrix, which is Hessenberg. We can then use A-orbits of dimension $\binom{n}{2}$ to construct polarizations of dense, open submanifolds of regular adjoint orbits. Hence, the Gelfand-Zeitlin system is completely integrable on each regular adjoint orbit (Theorem 3.36 in [9]).

Our goal is to give a complete description of the A-orbit structure of $\mathfrak{gl}(n)^{sreg}$. It follows from the Poisson commutativity of the algebra $J(\mathfrak{gl}(n))$ in (2.3) that the fibres $\mathfrak{gl}(n)_c$ are A-stable. Whence, the fibres $\mathfrak{gl}(n)_c^{sreg}$ are A-stable. Moreover, Theorem 3.12 in [9] implies

that the A-orbits in $\mathfrak{gl}(n)^{sreg}$ are the irreducible components of the fibres $\mathfrak{gl}(n)_c^{sreg}$. From this it follows that

$$\text{there are only finitely many } A\text{-orbits in the fibre } \mathfrak{gl}(n)_c^{sreg}.$$

In this paper, we describe the A-orbit structure of an arbitrary fibre $\mathfrak{gl}(n)_c^{sreg}$ and count the exact number of A-orbits in the fibre. This gives a complete description of the A-orbit structure of $\mathfrak{gl}(n)^{sreg}$.

Remark 2.6. Note that the collection of fibres of the map Φ is the same as the collection of fibres of the moment map for the A-action $x \to (f_{1,1}(x_1), f_{2,1}(x_2), \cdots, f_{n,n}(x))$. Thus, studying the action of A on the fibres of Φ is essentially studying the action of A on the fibres of the corresponding moment map. We use the map Φ instead of the moment map, since it is easier to describe the fibres of Φ.

For our purposes, it is convenient to have a more concrete characterization of strongly regular elements. (See Theorem 2.14 in [9].)

Proposition 2.7. *Let $x \in \mathfrak{gl}(n)$ and let $\mathfrak{z}_{\mathfrak{gl}(i)}(x_i)$ denote the centralizer in $\mathfrak{gl}(i)$ of x_i. Then x is strongly regular if and only if the following two conditions hold.*
(a) $x_i \in \mathfrak{gl}(i)$ is regular for all i, $1 \leq i \leq n$.
(b) $\mathfrak{z}_{\mathfrak{gl}(i-1)}(x_{i-1}) \cap \mathfrak{z}_{\mathfrak{gl}(i)}(x_i) = 0$ for all $2 \leq i \leq n$.

3. The action of A on generic matrices

For $x \in \mathfrak{gl}(i)$, let $\sigma(x)$ denote the spectrum of x, where x is viewed as an element of $\mathfrak{gl}(i)$. We consider the following Zariski open subset of regular semisimple elements of $\mathfrak{gl}(n)$

$$\mathfrak{gl}(n)_\Omega = \{x \in \mathfrak{gl}(n) |\ x_i \text{ is regular semisimple}, \sigma(x_{i-1}) \cap \sigma(x_i) = \emptyset,\ 2 \leq i \leq n\}. \tag{3.1}$$

Kostant and Wallach give a complete description of the action of A on $\mathfrak{gl}(n)_\Omega$. We give an example of a matrix in $\mathfrak{gl}(3)_\Omega$.

Example 3.1. Consider the matrix in $\mathfrak{gl}(3)$

$$X = \begin{bmatrix} 1 & 2 & 16 \\ 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}.$$

We can compute that X has eigenvalues $\sigma(X) = \{-3, 3, -2\}$ so that X is regular semisimple and that $\sigma(X_2) = \{2, -1\}$. Clearly $\sigma(X_1) = \{1\}$. Thus $X \in \mathfrak{gl}(3)_\Omega$.

We recall the notational convention introduced in (1.2). (If $c_i = (z_1, z_2, \cdots, z_i) \in \mathbb{C}^i$, then $p_{c_i}(t) = z_1 + z_2 t + \cdots + z_i t^{i-1} + t^i$.) Let $\Omega_n \subset \mathbb{C}^{\frac{n(n+1)}{2}}$ be the Zariski open subset consisting of $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$ with $c = (c_1, \cdots, c_i, \cdots, c_n)$ such that $p_{c_i}(t)$ has distinct roots and $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have no roots in common (remark 2.16 in [9]). It is easy to see that $\mathfrak{gl}(n)_\Omega = \bigcup_{c \in \Omega_n} \mathfrak{gl}(n)_c$.

Kostant and Wallach describe the A-orbit structure on $\mathfrak{gl}(n)_\Omega$ in Theorems 3.23 and 3.28 in [9]. We summarize the results of both of these theorems in one statement below.

Theorem 3.2. *The elements of $\mathfrak{gl}(n)_\Omega$ are strongly regular. If $c \in \Omega_n$, then $\mathfrak{gl}(n)_c = \mathfrak{gl}(n)_c^{sreg}$ is precisely one orbit under the action of the group A. Moreover, $\mathfrak{gl}(n)_c$ is a homogeneous space for a free, algebraic action of the torus $(\mathbb{C}^\times)^{\binom{n}{2}}$.*

We sketch the ideas behind one possible proof of Theorem 3.2 in the case of $\mathfrak{gl}(3)$. For complete proofs and a more thorough explanation, see either [9] or [3].

For $x \in \mathfrak{gl}(3)$ its A-orbit is

$$\mathrm{Ad}\left(\begin{bmatrix} z_1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} z_2 & & \\ & z_2 & \\ & & 1 \end{bmatrix}\begin{bmatrix} \multicolumn{2}{c}{\exp(tx_2)} & \\ & & 1 \end{bmatrix}\right) \cdot x, \tag{3.2}$$

where $z_1,\ z_2 \in \mathbb{C}^\times$ and $t \in \mathbb{C}$. (See equation (2.6).)

If we let $Z_i \subset GL(i)$ be the centralizer of x_i in $GL(i)$, we notice from (3.2) the action of A appears to push down to an action of $Z_1 \times Z_2$. For $x \in \mathfrak{gl}(3)_\Omega$, we should then expect to see an action of $(\mathbb{C}^\times)^3$ as realizing the action of A.

Working directly from the definition of the action of A in (3.2) is cumbersome. The action of Z_2 on x_2 for would be much easier to write down if x_2 were diagonal. For $x \in \mathfrak{gl}(3)_\Omega$, x_2 is not diagonal, but it is diagonalizable. So, we first diagonalize x_2 and then conjugate by the centralizer $Z_2 = (\mathbb{C}^\times)^2$. If $\gamma(x) \in GL(2)$ is such $(\mathrm{Ad}(\gamma(x)) \cdot x)_2$ is diagonal, then we can define an action of $(\mathbb{C}^\times)^3$ on $\mathfrak{gl}(3)_c$ for $c \in \Omega_3$ by

$$(z_1', z_2', z_3') \cdot x = \mathrm{Ad}\left(\begin{bmatrix} z_1' & & \\ & 1 & \\ & & 1 \end{bmatrix}\gamma(x)^{-1}\begin{bmatrix} z_2' & & \\ & z_3' & \\ & & 1 \end{bmatrix}\gamma(x)\right) \cdot x, \tag{3.3}$$

with $z_i' \in \mathbb{C}^\times$.

We can show (3.3) is a simply transitive algebraic group action on $\mathfrak{gl}(3)_c$ by explicit computation. Comparing (3.3) and (3.2), it is not hard to believe that the action of $(\mathbb{C}^\times)^3$ in (3.3) has the same orbits as the action of A on $\mathfrak{gl}(3)_c$. To prove this precisely, one needs to see that $\mathfrak{gl}(3)_c^{sreg} = \mathfrak{gl}(3)_c$. This can be proven by computing the tangent space to the action of $(\mathbb{C}^\times)^3$ in (3.3) and showing that it is same as the subspace V_x in (2.7), or by appealing to Theorem 2.17 in [9]. The fact that $\mathfrak{gl}(3)_c$ is one A-orbit then follows easily by applying Theorem 3.12 in [9].

This line of argument is not the one used in [9] to prove Theorem 3.2. The ideas here go back to a preliminary approach by Kostant and Wallach. However, it is this method that generalizes to describe all orbits of A in $\mathfrak{gl}(n)^{sreg}$. We describe the general construction in the next section.

4. Constructing non-generic A-orbits

4.1. **Overview.** In the next three sections, we classify A-orbits in $\mathfrak{gl}(n)^{sreg}$ by determining the A-orbit structure of an arbitrary fibre $\mathfrak{gl}(n)^{sreg}_c$. Let $c_i \in \mathbb{C}^i$ and $p_{c_i}(t) = (t-\lambda_1)^{n_1}\cdots(t-\lambda_r)^{n_r}$ with $\lambda_j \neq \lambda_k$ for $j \neq k$ (see 1.2). To study the action of A on $\mathfrak{gl}(n)_c$ with $c = (c_1, \cdots, c_i, c_{i+1}, \cdots, c_n) \in \mathbb{C}^1 \times \cdots \times \mathbb{C}^i \times \mathbb{C}^{i+1} \times \cdots \times \mathbb{C}^n = \mathbb{C}^{\frac{n(n+1)}{2}}$, we consider elements of $\mathfrak{gl}(i+1)$ of the form

$$\left[\begin{array}{ccccc} \begin{bmatrix} \lambda_1 & 1 & \cdots & 0 \\ 0 & \lambda_1 & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_1 \end{bmatrix} & & 0 & & \begin{matrix} y_{1,1} \\ \vdots \\ \vdots \\ y_{1,n_1} \end{matrix} \\ & \ddots & & & \vdots \\ 0 & & \begin{bmatrix} \lambda_r & 1 & \cdots & 0 \\ 0 & \lambda_r & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & \lambda_r \end{bmatrix} & & \begin{matrix} y_{r,1} \\ \vdots \\ \vdots \\ y_{r,n_r} \end{matrix} \\ z_{1,1} \ \cdots \ \cdots \ z_{1,n_1} & \cdots & z_{r,1} \ \cdots \ \cdots \ z_{r,n_r} & & w \end{array}\right] \tag{4.1}$$

with characteristic polynomial $p_{c_{i+1}}(t)$.

To avoid ambiguity, it is necessary to order the Jordan blocks of the $i \times i$ cutoff of the matrix in (4.1). To do this, we introduce a lexicographical ordering on $\mathbb{C}$ defined as follows. Let $z_1, z_2 \in \mathbb{C}$, we say that $z_1 > z_2$ if and only if $Rez_1 > Rez_2$ or if $Rez_1 = Rez_2$, then $Imz_1 > Imz_2$.

Definition 4.1. Let $c_i \in \mathbb{C}^i$ be such that $p_{c_i}(t) = (t-\lambda_1)^{n_1}\cdots(t-\lambda_r)^{n_r}$ with $\lambda_j \neq \lambda_k$ (as in (1.2)) and let $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ in the lexicographical ordering on $\mathbb{C}$. For $c_{i+1} \in \mathbb{C}^{i+1}$, we define $\Xi^i_{c_i, c_{i+1}}$ as the set of elements $x \in \mathfrak{gl}(i+1)$ of the form (4.1) whose characteristic polynomial is $p_{c_{i+1}}(t)$. We refer to $\Xi^i_{c_i, c_{i+1}}$ as the solution variety at level i.

We know from Theorem 2.5 that $\Xi^i_{c_i, c_{i+1}}$ is non-empty for any $c_i \in \mathbb{C}^i$ and any $c_{i+1} \in \mathbb{C}^{i+1}$. Let us denote the regular Jordan form which is the $i \times i$ cutoff of the matrix in (4.1) by J. Let Z_i denote the centralizer of J in $GL(i)$. As J is regular, Z_i is a connected, abelian algebraic group (see Proposition 14 in [8]). Z_i acts algebraically on the solution variety $\Xi^i_{c_i,c_{i+1}}$ by conjugation. In the remainder of section 4, we give a bijection between A-orbits in $\mathfrak{gl}(n)^{sreg}_c$ and free $Z_1 \times \cdots \times Z_{n-1}$ orbits on $\Xi^1_{c_1,c_2} \times \cdots \times \Xi^{n-1}_{c_{n-1},c_n}$. In Section 5, we will classify the Z_i-orbits on $\Xi^i_{c_i, c_{i+1}}$ using combinatorial data of the tuple $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$. We will then have a complete picture of the A action on $\mathfrak{gl}(n)^{sreg}_c$.

We now give a brief outline of the construction, which gives the bijection between A-orbits in $\mathfrak{gl}(n)^{sreg}_c$ and $Z_1 \times \cdots \times Z_{n-1}$ orbits in $\Xi^1_{c_1,c_2} \times \cdots \times \Xi^{n-1}_{c_{n-1},c_n}$. This construction not

only describes A-orbits in $\mathfrak{gl}(n)_c^{sreg}$, but all A-orbits in the larger set $\mathfrak{gl}(n)_c \cap S$, where S is the Zariski open subset of $\mathfrak{gl}(n)$ consisting of elements x whose cutoffs x_i for $1 \leq i \leq n-1$ are regular. We know by Proposition 2.7 (a) that $\mathfrak{gl}(n)_c^{sreg} \subset \mathfrak{gl}(n)_c \cap S$, and it is in general a proper subset. (See Example 5.4 below.)

The construction proceeds as follows. For $1 \leq i \leq n-2$, we choose a Z_i-orbit $\mathcal{O}^i_{a_i} \in \Xi^i_{c_i, c_{i+1}}$ consisting of regular elements of $\mathfrak{gl}(i+1)$. For $i = n-1$, we choose any orbit $\mathcal{O}^{n-1}_{a_{n-1}}$ of Z_{n-1} in $\Xi^{n-1}_{c_{n-1}, c_n}$. Then we define a morphism

$$\Gamma_n^{a_1, a_2, \cdots, a_{n-1}} : \mathcal{O}^1_{a_1} \times \cdots \times \mathcal{O}^{n-1}_{a_{n-1}} \to \mathfrak{gl}(n)_c \cap S.$$

by

$$\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}(x_1, \cdots, x_{n-1}) = \mathrm{Ad}(g_{1,2}(x_1)^{-1} g_{2,3}(x_2)^{-1} \cdots g_{n-2,n-1}(x_{n-2})^{-1}) x_{n-1}. \tag{4.2}$$

where $g_{i,i+1}(x_i)$ conjugates x_i into Jordan canonical form (with eigenvalues in descreasing lexicographical order). We denote the image of the morphism $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ by $Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$.

Theorem 4.2. *Every A-orbit in $\mathfrak{gl}(n)_c \cap S$ is of the form $Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ for some choice of orbits $\mathcal{O}^i_{a_i} \subset \Xi^i_{c_i, c_{i+1}}$ with $\mathcal{O}^i_{a_i}$ consisting of regular elements of $\mathfrak{gl}(i+1)$ for $1 \leq i \leq n-2$.*

In section 4.3, we prove Theorem 4.2 for A-orbits in $\mathfrak{gl}(n)_c^{sreg}$ (see Theorem 4.9). In section 4.4, we establish the results needed to prove Thoerem 4.2 for $\mathfrak{gl}(n)_c \cap S$.

4.2. **Definition and properties of the $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ maps.** We first define the map $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ only for Z_i-orbits $\mathcal{O}^i_{a_i} \subset \Xi^i_{c_i, c_{i+1}}$ on which Z_i acts freely. To define the map $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$, we must define a morphism $\mathcal{O}^i_{a_i} \to GL(i+1)$ which sends $y \to g_{i,i+1}(y)$, where $g_{i,i+1}(y)$ conjugates y into Jordan form with eigenvalues in decreasing lexicographical order. Since Z_i acts freely on $\mathcal{O}^i_{a_i}$, we can identify $\mathcal{O}^i_{a_i} \simeq Z_i$ as algebraic varieties. Let x_{a_i} be an arbitrary choice of base point for the orbit $\mathcal{O}^i_{a_i}$, i.e. $\mathcal{O}^i_{a_i} = \mathrm{Ad}(Z_i) \cdot x_{a_i}$. We choose an element $g_{i,i+1}(x_{a_i}) \in GL(i+1)$ that conjugates the base point x_{a_i} into Jordan form (with eigenvalues in decreasing lexicographical order). For $y = \mathrm{Ad}(k_i) \cdot x_{a_i}$, with $k_i \in Z_i$, we define

$$g_{i,i+1}(y) = g_{i,i+1}(x_{a_i}) k_i^{-1}. \tag{4.3}$$

For each choice of orbit $\mathcal{O}^i_{a_i} \subset \Xi^i_{c_i, c_{i+1}}$ for $1 \leq i \leq n-1$, we define a morphism $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}} : Z_1 \times \cdots \times Z_{n-1} \to \mathfrak{gl}(n)$,

(4.4)

$$\Gamma_n^{a_1, \cdots, a_{n-1}}(k_1, \cdots, k_{n-1}) = \mathrm{Ad}(k_1 g_{1,2}(x_{a_1})^{-1} k_2 g_{2,3}(x_{a_2})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}}.$$

We want to give a more intrinsic characterization of the image of the morphism $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$.

Proposition 4.3. *The set $Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}} \subset \mathfrak{gl}(n)_c \cap S$ and is equal to*

$$Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}} = \{x \in \mathfrak{gl}(n) |\ x_{i+1} \in \mathrm{Ad}(GL(i)) \cdot x_{a_i}, \ \textit{for all } 1 \leq i \leq n-1\}. \tag{4.5}$$

Thus, $Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ is a quasi-affine subvariety of $\mathfrak{gl}(n)$.

The following simple observation is useful in proving Proposition 4.3.

Remark 4.4. Let $x \in \mathfrak{gl}(n)_c \cap S$, and suppose that $g \in GL(i)$ is such that $\mathrm{Ad}(g) \cdot x = \mathrm{Ad}(g) \cdot x_i$ is in Jordan canonical form with eigenvalues in decreasing lexicographical order for $1 \leq i \leq n-1$. Then $[\mathrm{Ad}(g) \cdot x]_{i+1} = \mathrm{Ad}(g) \cdot x_{i+1} \in \Xi^i_{c_i,\, c_{i+1}}$.

Proof of Proposition 4.3. For ease of notation, let us denote the set on the RHS of (4.5) by T. We note $T \subset \mathfrak{gl}(n)_c \cap S$. Indeed, let $Y \in T$. Then $Y_{i+1} \in \mathrm{Ad}(GL(i)) \cdot x_{a_i}$ for $1 \leq i \leq n-1$. Since $x_{a_i} \in \Xi^i_{c_i,c_{i+1}}$, Y_{i+1} has characteristic polynomial $p_{c_{i+1}}(t)$. Also note that for $1 \leq i \leq n-2$, x_{a_i} is regular, and hence Y_{i+1} is regular for $1 \leq i \leq n-2$. Lastly, using the fact that $k_1 \in GL(1) = Z_1$ centralizes the $(1,1)$ entry of $x_{a_1} \in \Xi^1_{c_1,c_2}$, it follows that the $(1,1)$ entry of Y is given by $c_1 \in \mathbb{C}$.

The inclusion $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} \subset T$ is clear from the definition of $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ in (4.4). To see the opposite inclusion we use induction. Let $y \in T$. Then $y_2 \in \mathrm{Ad}(GL(1)) \cdot x_{a_1} = \mathcal{O}^1_{a_1}$, since $Z_1 = GL(1)$. Thus, there exists a $k_1 \in Z_1$ such that $y_2 = \mathrm{Ad}(k_1) \cdot x_{a_1}$. It follows that

$$z_2 = [\mathrm{Ad}(g_{1,2}(x_{a_1}))\mathrm{Ad}(k_1^{-1}) \cdot y]_3 = [\mathrm{Ad}(g_{1,2}(x_{a_1}))\mathrm{Ad}(k_1^{-1}) \cdot y_3] \in \Xi^2_{c_2,c_3}.$$

But $y_3 \in \mathrm{Ad}(GL(2)) \cdot x_{a_2}$, so that $z_2 \in \Xi^2_{c_2,c_3} \cap \mathrm{Ad}(GL(2)) \cdot x_{a_2}$. From which it follows easily that $z_2 \in \mathcal{O}^2_{a_2}$. Thus, there exists a $k_2 \in Z_2$ such that

$$[\mathrm{Ad}(g_{2,3}(x_{a_2}))\mathrm{Ad}(k_2^{-1})\mathrm{Ad}(g_{1,2}(x_{a_1}))\mathrm{Ad}(k_1^{-1}) \cdot y]_4 \in \Xi^3_{c_3,c_4}.$$

This completes the first two steps of the induction. We now assume that there exist $k_1, \cdots, k_{j-1} \in Z_1, \cdots, Z_{j-1}$, respectively such that

$$z_j = [\mathrm{Ad}(g_{j-1,j}(x_{a_{j-1}}))\mathrm{Ad}(k_{j-1}^{-1}) \cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\mathrm{Ad}(k_1^{-1}) \cdot y]_{j+1} \in \Xi^j_{c_j,c_{j+1}}. \tag{4.6}$$

Since $y_{j+1} \in \mathrm{Ad}(GL(j)) \cdot x_{a_j}$, it follows that $z_j \in \Xi^j_{c_j,c_{j+1}} \cap \mathrm{Ad}(GL(j)) \cdot x_{a_j}$. As above, it follows that $z_j \in \mathcal{O}^j_{a_j}$, so that there exists an element $k_j \in K_j$ such that

$$[\mathrm{Ad}(g_{j,j+1}(x_{a_j}))\mathrm{Ad}(k_j^{-1})\mathrm{Ad}(g_{j-1,j}(x_{a_{j-1}}))\mathrm{Ad}(k_{j-1}^{-1}) \cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\mathrm{Ad}(k_1^{-1}) \cdot y]_{j+2} \in \Xi^{j+1}_{c_{j+1},c_{j+2}}.$$

We have made use of Remark 4.4 throughout. By induction, we conclude that there exist $k_1, \cdots, k_{n-1} \in Z_1, \cdots, Z_{n-1}$ respectively so that

$$x_{a_{n-1}} = \mathrm{Ad}(k_{n-1}^{-1})\mathrm{Ad}(g_{n-2,n-1}(x_{a_{n-1}}))\mathrm{Ad}(k_{n-2}^{-1}) \cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\mathrm{Ad}(k_1^{-1}) \cdot y.$$

From which it follows that $y = \Gamma_n^{a_1,\cdots,a_{n-1}}(k_1, \cdots, k_{n-1})$.

To see the final statement of the proposition, we observe T is a Zariski locally closed subset of $\mathfrak{gl}(n)$. Indeed, the set $U_i = \{x | x_{i+1} \in \mathrm{Ad}(GL(i)) \cdot x_{a_i}\}$ is locally closed, since it is the preimage of the orbit $\mathrm{Ad}(GL(i)) \cdot x_{a_i} \subset \mathfrak{gl}(i+1)$ under the projection morphism $\pi_{i+1}(x) = x_{i+1}$. The set $T = U_1 \cap \cdots \cap U_{n-1}$ is locally closed.

Q.E.D.

Remark 4.5. From Proposition 4.3 it follows that the set $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ depends only on the orbits $\mathcal{O}^i_{a_i}$ for $1 \leq i \leq n-1$, and is thus independent of the choices involved in defining the map $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ in (4.4).

4.3. $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ **and A-orbits in $\mathfrak{gl}(n)_c^{sreg}$.** In this section, we show that the image of the morphism $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ is an A-orbit in $\mathfrak{gl}(n)_c^{sreg}$. The first step is to see $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ is smooth variety.

Theorem 4.6. *The morphism*

$$\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} : Z_1 \times \cdots \times Z_{n-1} \to \mathfrak{gl}(n)_c \cap S$$

is an isomorphism onto its image. Hence, $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ is a smooth, irreducible subvariety of $\mathfrak{gl}(n)$ of dimension $\binom{n}{2}$.

Proof. We explicitly construct an inverse Ψ to $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ and show that $\Psi : Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} \to Z_1 \times \cdots \times Z_{n-1}$ is a morphism. Specifically, we show that there exist morphisms $\psi_i : Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} \to Z_i$ for $1 \le i \le n-1$ so that the morphism

$$\Psi = (\psi_1, \cdots, \psi_{n-1}) : Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} \to Z_1 \times \cdots \times Z_{n-1} \tag{4.7}$$

is an inverse to $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$. The morphisms ψ_i are constructed inductively.

Given $y \in Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$, $y_2 \in \mathcal{O}^1_{a_1} \subset \Xi^1_{c_1,c_2}$ by Proposition 4.3. Thus, $y_2 = \mathrm{Ad}(k_1)\cdot x_{a_1}$ for a unique k_1 in Z_1. The map $\mathcal{O}^1_{a_1} \to Z_1$ given by $\mathrm{Ad}(k_1)\cdot x_{a_1} \to k_1$ is an isomorphism of smooth affine varieties. Hence, the map $\psi_1(y) = k_1$ is a morphism.

Arguing as in the proof of Proposition 4.3, suppose that we have defined morphisms $\psi_1, \cdots, \psi_{j-1}$, with $\psi_i : Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} \to Z_i$ for $1 \le i \le j-1$. Then the function $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} \to \mathcal{O}^j_{a_j}$ given by equation (4.6),

$$y \to [\mathrm{Ad}(g_{j-1,j}(x_{a_{j-1}}))\mathrm{Ad}(\psi_{j-1}(y)^{-1})\cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\mathrm{Ad}(\psi_1(y)^{-1})\cdot y]_{j+1}$$

is a morphism. We can then define a morphism $\psi_j : Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} \to Z_j$ given by $\psi_j(y) = k_j$, where k_j is the unique element of Z_j such that

$$\mathrm{Ad}(k_j)\cdot x_{a_j} = [\mathrm{Ad}(g_{j-1,j}(x_{a_{j-1}}))\mathrm{Ad}(\psi_{j-1}(y)^{-1})\cdots \mathrm{Ad}(g_{1,2}(x_{a_1}))\mathrm{Ad}(\psi_1(y)^{-1})\cdot y]_{j+1}. \tag{4.8}$$

This completes the induction.

Now, we need to see that the map Ψ is an inverse to $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$. The fact that $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}(\psi_1(y), \cdots, \psi_{n-1}(y)) = y$ follows exactly as in the proof of the inclusion $T \subset Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ in Proposition 4.3.

Finally, we show that $\Psi(\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}(k_1, \cdots, k_{n-1})) = (k_1, \cdots, k_{n-1})$. We make the following observation. Consider the element

$$\mathrm{Ad}(k_j g_{j,j+1}(x_{a_j})^{-1}\cdots g_{n-2,n-1}(x_{a_{n-2}})^{-1}k_{n-1})\cdot x_{a_{n-1}}.$$

The $(j+1)\times(j+1)$ cutoff of this element is equal to $k_j \cdot x_{a_j}$. Using this fact with $j = 1$, we have $\psi_1(y) = k_1$. Assume that we have $\psi_2(y) = k_1, \cdots, \psi_l(y) = k_l$ for $2 \le l \le j-1$. Using the definition of ψ_j in (4.8), we obtain

$$\mathrm{Ad}(\psi_j(y))\cdot x_{a_j} = [\mathrm{Ad}(k_j)\mathrm{Ad}(g_{j,j+1}(x_{a_j})^{-1}\cdots g_{n-2,n-1}(x_{a_{n-2}})^{-1}k_{n-1})x_{a_{n-1}}]_{j+1} = \mathrm{Ad}(k_j)\cdot x_{a_j}.$$

Thus, by induction $\Psi \circ \Gamma_n^{a_1,a_2,\cdots,a_{n-1}} = id$. Hence, Ψ is a regular inverse to the map $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ and Ψ is an isomorphism of varieties.

Q.E.D.

$Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ is a smooth, irreducible quasi-affine subvariety of $\mathfrak{gl}(n)$. Thus, $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ has the structure of a connected analytic submanifold of $\mathfrak{gl}(n)$, and $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ is an analytic isomorphism. We now show that the action of the analytic group A preserves the submanifold $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$.

Proposition 4.7. *The action of A on $\mathfrak{gl}(n)$ preserves the submanifolds $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$.*

Proof. We recall that the action of A on $\mathfrak{gl}(n)$ is given by the composition of the flows in (2.6) in any order. (See Remark 2.3.) Thus, to see that the action of A preserves $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ it suffices to see that the action of $\mathbb{C}$ in (2.6) preserves $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ for any $1 \leq i \leq n-1$ and any $1 \leq j \leq i$. This can be seen easily using Proposition 4.3. Indeed, suppose that $x \in Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$. Then by Proposition 4.3, $x_{k+1} \in \mathrm{Ad}(GL(k)) \cdot x_{a_k}$ for any $1 \leq k \leq n-1$. Now we consider $\mathrm{Ad}(\exp(tjx_i^{j-1})) \cdot x$ as in (2.6) with $t \in \mathbb{C}$ fixed. For ease of notation let $h = \exp(tjx_i^{j-1}) \in GL(i)$. We claim $(\mathrm{Ad}(h) \cdot x)_{k+1} \in \mathrm{Ad}(GL(k)) \cdot x_{a_k}$ for $1 \leq k \leq n-1$. We consider two cases. Suppose $k \geq i$ and consider $(\mathrm{Ad}(h) \cdot x)_{k+1}$. We have $(\mathrm{Ad}(h) \cdot x)_{k+1} = \mathrm{Ad}(h) \cdot x_{k+1}$. But $x_{k+1} \in \mathrm{Ad}(GL(k)) \cdot x_{a_k}$, so that $\mathrm{Ad}(h) \cdot x_{k+1} \in \mathrm{Ad}(GL(k)) \cdot x_{a_k}$, as $GL(i) \subset GL(k)$. Next, we suppose that $k < i$, so that $k+1 \leq i$. Since $h \in GL(i)$ centralizes x_i,

$$(\mathrm{Ad}(h)x)_{k+1} = (\mathrm{Ad}(h)(x_i))_{k+1} = (x_i)_{k+1} = x_{k+1} \in \mathrm{Ad}(GL(k)) \cdot x_{a_k}.$$

By Proposition 4.3 $\mathrm{Ad}(h) \cdot x \in Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$. This completes the proof.

Q.E.D.

Before stating the main theorem of this section, we need to state a technical result about the action of Z_i on the solution varieties $\Xi^i_{c_i,\,c_{i+1}}$. This result will be proven independently of the following theorem in section 4.4.

Lemma 4.8. *For $x \in \Xi^i_{c_i,\,c_{i+1}}$, the isotropy group of x under the action of Z_i, $Stab(x)$, is a connected algebraic group.*

Thus, given an orbit of Z_i, $\mathcal{O} \subset \Xi^i_{c_i,\,c_{i+1}}$

$$\dim(\mathcal{O}) = i \text{ if and only if } Z_i \text{ acts freely on } \mathcal{O}. \tag{4.9}$$

We are now ready to prove the main theorem of this section.

Theorem 4.9. *The submanifold $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} \subset \mathfrak{gl}(n)_c \cap S$ is a single A-orbit in $\mathfrak{gl}(n)_c^{sreg}$. Moreover every A-orbit in $\mathfrak{gl}(n)_c^{sreg}$ is of the form $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ for some choice of free Z_i-orbits $\mathcal{O}^i_{a_i} \subset \Xi^i_{c_i,\,c_{i+1}}$ with $\mathcal{O}^i_{a_i} \subset \mathfrak{gl}(i+1)^{reg}$, for $1 \leq i \leq n-1$.*

Proof. First, we show that $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ is an A-orbit. For this, we need to describe the tangent space $T_y(Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}) = (d\Gamma_n^{a_1,a_2,\cdots,a_{n-1}})_{\underline{k}}$, where $\underline{k} = (k_1, \cdots, k_{n-1}) \in$

$Z_1 \times \cdots \times Z_{n-1}$ and $y = \Gamma_n^{a_1,a_2,\cdots,a_{n-1}}(\underline{k})$. Let $\{\alpha_{i1}, \cdots, \alpha_{ii}\}$ be a basis for $Lie(Z_i) = \mathfrak{z}_i$. Working analytically, we compute

(4.10)
$$(d\Gamma_n^{a_1,a_2,\cdots,a_{n-1}})_{\underline{k}}(0, \cdots, \alpha_{ij}, \cdots, 0) = \frac{d}{dt}|_{t=0}\ \Gamma_n^{a_1,a_2,\cdots,a_{n-1}}(k_1, \cdots, k_i \exp(t\alpha_{ij}), \cdots, k_{n-1}),$$

for $1 \leq j \leq i$. Using the definition of the morphism $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ the RHS of (4.10) becomes

(4.11)
$$\frac{d}{dt}|_{t=0}\mathrm{Ad}(k_1 g_{1,2}(x_{a_1})^{-1} \cdots k_i \exp(t\alpha_{ij}) g_{i,i+1}(x_{a_i})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}) x_{a_{n-1}}.$$

Let

$$l_i = k_1 g_{1,2}(x_{a_1})^{-1} \cdots k_i \text{ and let } h_i = g_{i,i+1}(x_{a_i})^{-1} \cdots k_{n-2} g_{n-2,n-1}(x_{a_{n-2}})^{-1} k_{n-1}. \tag{4.12}$$

Then we can write (4.11) as

$$\frac{d}{dt}|_{t=0}\mathrm{Ad}(l_i \exp(t\alpha_{ij}) h_i) \cdot x_{a_{n-1}},$$

which has differential

$$\mathrm{ad}(\mathrm{Ad}(l_i) \cdot \alpha_{ij}) \cdot (\mathrm{Ad}(l_i h_i) \cdot x_{a_{n-1}}). \tag{4.13}$$

By definition of the element $l_i \in GL(i)$, the $i \times i$ cutoff of $\mathrm{Ad}(l_i^{-1}) \cdot y = \mathrm{Ad}(l_i^{-1}) \cdot y_i$ is in Jordan form (with eigenvalues in decreasing lexicographical order). Hence elements of the form $\mathrm{Ad}(l_i) \cdot \alpha_{ij} = \gamma_{ij}$ for $1 \leq j \leq i$ form a basis for $\mathfrak{z}_{\mathfrak{gl}(i)}(y_i)$. Since $\mathrm{Ad}(l_i h_i) \cdot x_{a_{n-1}} = y$, (4.13) implies the image of $d(\Gamma_n^{a_1,a_2,\cdots,a_{n-1}})_{\underline{k}}$ is

(4.14)
$$Im(d\Gamma_n^{a_1,a_2,\cdots,a_{n-1}})_{\underline{k}} = span\{\partial_y^{[\gamma_{i,j},y]},\ 1 \leq i \leq n-1, 1 \leq j \leq i\} = T_y(Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}).$$

We recall equation (2.7),

$$T_y(A \cdot y) = span\{\partial_y^{[z,y]} | z \in Z_y\} := V_y.$$

Now, $y \in Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ has the property that y_i is regular for all $i \leq n-1$, so that $\mathfrak{z}_{\mathfrak{gl}(i)}(y_i)$ has basis $\{Id_i, y_i, \cdots, y_i^{i-1}\}$ (see [8], pg 382). Thus,

$$T_y(Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}) = span\{\partial_y^{[z,y]} | z \in Z_y\} = V_y. \tag{4.15}$$

Equation (4.15) gives

$$\dim V_y = \dim(A \cdot y) = \binom{n}{2}, \tag{4.16}$$

which implies $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}} \subset \mathfrak{gl}(n)_c^{sreg}$. By Proposition 4.7, A acts on $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$. We claim that the action of A is transitive on $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$. Indeed, given an A-orbit $A \cdot y$ with $y \in Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$, $A \cdot y \subset Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ is a submanifold of the same dimension as $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ by (4.16), and thus must be open. The action of A is then clearly transitive on $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$, as $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ is connected.

We now show that every A-orbit in $\mathfrak{gl}(n)_c^{sreg}$ is obtained in this manner. For $x \in \mathfrak{gl}(n)_c^{sreg}$, by part (a) of Proposition 2.7 and Remark 4.4 there exists a matrix $g_i \in GL(i)$ such that $z_i = Ad(g_i) \cdot x_{i+1} \in \Xi^i_{c_i, c_{i+1}}$ and z_i is regular for each $1 \leq i \leq n-1$. Thus $z_i \in \mathcal{O}^i_{a_i}$, with $\mathcal{O}^i_{a_i}$ an orbit of Z_i in $\Xi^i_{c_i, c_{i+1}}$ consisting of regular elements of $\mathfrak{gl}(i+1)$. We claim that Z_i must act freely on $\mathcal{O}^i_{a_i}$. We suppose to the contrary that $Stab(x_{a_i})$ is non-trivial. Lemma 4.8 gives that $\dim(Stab(x_{a_i})) \geq 1$. But, this implies $\dim(Z_{GL(i)}(x_i) \cap Z_{GL(i+1)}(x_{i+1})) \geq 1$, contradicting part (b) of Proposition 2.7. By Proposition 4.3 $x \in Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ for some choice of free Z_i-orbits $\mathcal{O}^i_{a_i} \subset \Xi^i_{c_i, c_{i+1}}$. This completes the proof of the theorem.

Q.E.D.

Remark 4.10. Let $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ be defined using Z_i-orbits, $\mathcal{O}^i_{a_i}$ and let $\Gamma_{n-1}^{\widetilde{a_1}, \widetilde{a_2}, \cdots, \widetilde{a_{n-1}}}$ be defined using Z_i-orbits $\mathcal{O}^i_{\widetilde{a_i}} = \mathrm{Ad}(Z_i) \cdot x_{\widetilde{a_i}}$, where for some i, $1 \leq i \leq n-1$, $\mathcal{O}^i_{a_i} \cap \mathcal{O}^i_{\widetilde{a_i}} = \emptyset$. Then it follows from Proposition 4.3 that the A-orbits $Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ and $Im\Gamma_{n-1}^{\widetilde{a_1}, \widetilde{a_2}, \cdots, \widetilde{a_{n-1}}}$ are distinct. Indeed, suppose to the contrary that $y \in Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}} \cap Im\Gamma_{n-1}^{\widetilde{a_1}, \widetilde{a_2}, \cdots, \widetilde{a_{n-1}}}$. By Proposition 4.3, we have $y_{i+1} \in \mathrm{Ad}(GL(i)) \cdot x_{a_i} \cap \mathrm{Ad}(GL(i)) \cdot x_{\widetilde{a_i}}$. This implies that there exists $h \in GL(i)$ such that $\mathrm{Ad}(h) \cdot x_{a_i} = x_{\widetilde{a_i}}$. Since $x_{a_i}, x_{\widetilde{a_i}} \in \Xi^i_{c_i, c_{i+1}}$, the previous equation forces $h \in Z_i$, which implies $\mathcal{O}^i_{a_i} = \mathcal{O}^i_{\widetilde{a_i}}$, a contradiction. We have thus established a bijection between free $Z_1 \times \cdots \times Z_{n-1}$ orbits on the product of solution varieties $\Xi^1_{c_1, c_2} \times \cdots \cdots \times \Xi^{n-1}_{c_{n-1}, c_n}$ and A-orbits in $\mathfrak{gl}(n)_c^{sreg}$.

On the subvariety $Im\Gamma_n^{a_1, \cdots, a_{n-1}}$, we have a free and transitive algebraic action of the algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$. This action is defined by the following formula.

(4.17)
$$\text{If } (\Gamma_n^{a_1, a_2, \cdots, a_{n-1}})^{-1}(y) = (k_1, \cdots, k_{n-1}), \text{ then } (k_1', \cdots, k_{n-1}') \cdot y = \Gamma_n^{a_1, a_2, \cdots, a_{n-1}}(k_1' k_1, \cdots, k_{n-1}' k_{n-1}).$$

Remark 4.11. The action in (4.17) is the generalization of the action of $(\mathbb{C}^\times)^3$ in (3.3) to the non-generic case.

Thus, the A-orbit $Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ is the orbit of an algebraic group acting on a quasi-affine variety. We now show that $Z = Z_1 \times \cdots \times Z_{n-1}$ acts algebraically on the fibre $\mathfrak{gl}(n)_c^{sreg}$. By Theorem 3.12 in [9] the A-orbits in $\mathfrak{gl}(n)_c^{sreg}$ are the irreducible components of $\mathfrak{gl}(n)_c^{sreg}$. Since they are disjoint, these components are both open and closed in $\mathfrak{gl}(n)_c^{sreg}$ (in the Zariski topology on $\mathfrak{gl}(n)_c^{sreg}$). Following [9], we index these components by $\mathfrak{gl}_{c,i}^{sreg}(n) = A \cdot x(i)$, with $x(i) \in \mathfrak{gl}(n)_c^{sreg}$. Now, we have morphisms $\phi_i : Z \times \mathfrak{gl}_{c,i}^{sreg}(n) \to \mathfrak{gl}_c^{sreg}(n)$ given by the action of Z on $Im\Gamma_n^{a_1, \cdots, a_{n-1}}$. The sets $Z \times \mathfrak{gl}_{c,i}^{sreg}(n)$ are (Zariski) open in the product $Z \times \mathfrak{gl}(n)_c^{sreg}$ and are disjoint. Thus, the morphisms ϕ_i glue to a unique morphism

$$\Phi : Z \times \mathfrak{gl}(n)_c^{sreg} \to \mathfrak{gl}(n)_c^{sreg} \text{ such that } \Phi|_{Z \times \mathfrak{gl}_{c,i}^{sreg}(n)} = \phi_i.$$

The morphism Φ defines an algebraic action of the group Z on $\mathfrak{gl}(n)_c^{sreg}$ whose orbits are the orbits of A in $\mathfrak{gl}(n)_c^{sreg}$. We have thus proven the following theorem.

Theorem 4.12. *Let $x \in \mathfrak{gl}(n)_c^{sreg}$ be arbitrary and let Z_i be the centralizer in $GL(i)$ of the Jordan form of x_i (with eigenvalues in decreasing lexicographical order). On $\mathfrak{gl}(n)_c^{sreg}$ the orbits of the group A are orbits of a free algebraic action of the connected abelian algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$.*

We end this section with a result that will be of great use in section 5 where we count the number of A-orbits in the fibre $\mathfrak{gl}(n)_c^{sreg}$.

It turns out that the condition in Theorem 4.9 that $\mathcal{O}_{a_i}^i \subset \mathfrak{gl}(i+1)^{reg}$ is superfluous.

Theorem 4.13. *If $\mathcal{O}_{a_i}^i \subset \Xi_{c_i,\, c_{i+1}}^i$ is a free Z_i-orbit, then $\mathcal{O}_{a_i}^i \subset \mathfrak{gl}(i+1)^{reg}$.*

Proof. Let $c = (c_1, c_2, \cdots, c_j, c_{j+1}, \cdots, c_n) \in \mathbb{C}^{\frac{n(n+1)}{2}}$, with $c_j \in \mathbb{C}^j$ be given. By Theorem 2.5, there is a unique upper Hessenberg matrix $h \in \mathfrak{gl}(n)_c^{sreg}$. This implies that for any j, $1 \leq j \leq n-1$, there exists a $g_j \in GL(j)$ such that $(\mathrm{Ad}(g_j) \cdot h)_{j+1} \in \Xi_{c_j, c_{j+1}}^j$ by Remark 4.4. Thus, $\mathrm{Ad}(g_j) \cdot h_{j+1} \in Z_j \cdot x_{a_j} = \mathcal{O}_{a_j}^j$ for some $x_{a_j} \in \Xi_{c_j, c_{j+1}}^j$. But $h \in \mathfrak{gl}(n)^{sreg}$ and therefore h_{j+1} is regular by part (a) of Proposition 2.7, which implies that $\mathcal{O}_{a_j}^j \subset \mathfrak{gl}(j+1)^{reg}$. Also, by part (b) of Proposition 2.7, Z_j acts freely on $\mathcal{O}_{a_j}^j$, as in the proof of the last statement of Theorem 4.9. Thus, for any j, $1 \leq j \leq n-1$, there exists a free Z_j-orbit in $\Xi_{c_j, c_{j+1}}^j$ consisting of regular elements of $\mathfrak{gl}(j+1)$.

Now, let $\mathcal{O}_{a_i}^i \subset \Xi_{c_i,\, c_{i+1}}^i$ be any free Z_i-orbit. Now, we use the free Z_j-orbit $\mathcal{O}_{a_j}^j \subset \mathfrak{gl}(j+1)^{reg}$ as above for $1 \leq j \leq i-1$ and $\mathcal{O}_{a_i}^i$ to construct a morphism $\Gamma_i^{a_1, a_2, \cdots, a_i} : Z_1 \times \cdots \times Z_i \to \mathfrak{gl}(n)_c \cap S$. By Theorem 4.9, $Im\Gamma_i^{a_1, a_2, \cdots, a_i} \subset \mathfrak{gl}(i+1)^{sreg}$. Proposition 2.7 (a) then implies $Im\Gamma_i^{a_1, a_2, \cdots, a_i} \subset \mathfrak{gl}(i+1)^{reg}$. Since elements of $\mathcal{O}_{a_i}^i$ are conjugate to elements of $Im\Gamma_i^{a_1, a_2, \cdots, a_i}$, $\mathcal{O}_{a_i}^i \subset \mathfrak{gl}(i+1)^{reg}$. This completes the proof.

Q.E.D.

4.4. **A-orbits in $\mathfrak{gl}(n)_c \cap S$.** We now discuss how the construction in sections 4.2 and 4.3 can be generalized to describe A-orbits of dimension strictly less than $\binom{n}{2}$ in the Zariski open subset of the fibre $\mathfrak{gl}(n)_c \cap S$. In this case, it is more difficult to define the morphism $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ that appears in equation (4.2). The problem is that it is not clear how to define a morphism $\mathcal{O}_{a_i}^i \to GL(i+1)$ which sends $x \to g_{i,i+1}(x)$ where $\mathrm{Ad}(g_{i,i+1}(x)) \cdot x$ is in Jordan form (with eigenvalues in decreasing lexicographical order). This is not difficult in the strongly regular case, as we are dealing with free Z_i-orbits $\mathcal{O}_{a_i}^i \simeq Z_i$ so that $g_{i,i+1}(x)$ can be defined as in equation (4.3). The fortunate fact is that even for an orbit $\mathcal{O}_{a_i}^i \subset \Xi_{c_i,\, c_{i+1}}^i$ of dimension strictly less than i, there exists a connected, Zariski closed subgroup $K_i \subset Z_i$ with K_i acting freely on $\mathcal{O}_{a_i}^i \simeq K_i$. Therefore, we can mimic what we did in equation (4.3).

To prove this, we need to understand better the action of Z_i on $\Xi_{c_i,\, c_{i+1}}^i$. As in section 4.1, let $J = J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_r}$ be the $i \times i$ cutoff of the matrix in (4.1), where $J_{\lambda_j} \in \mathfrak{gl}(n_j)$ is the Jordan block corresponding to eigenvalue λ_j. We note since J is regular, Z_i is an

abelian connected algebraic group, which is the product of groups $\prod_{j=1}^{r} Z_{J_{\lambda_j}}$, where $Z_{J_{\lambda_j}}$ denotes the centralizer of J_{λ_j}. It is then easy to see that the action of Z_i is the diagonal action of the product $\prod_{j=1}^{r} Z_{J_{\lambda_j}}$ on the last column of $x \in \Xi^i_{c_i,c_{i+1}}$ and the dual action on the last row of x (see (4.1)). In other words, $Z_{J_{\lambda_j}}$ acts only on the columns and rows of x that contain the Jordan block J_{λ_j} (see (4.1)). This leads us to define an action of $Z_{J_{\lambda_j}}$ on $\mathbb{C}^{2n_j}$

$$z \cdot ([t_1, \cdots, t_{n_j}], [s_1, \cdots, s_{n_j}]^T) = ([t_1, \cdots, t_{n_j}] \cdot z^{-1}, z \cdot [s_1, \cdots, s_{n_j}]^T). \tag{4.18}$$

For $x \in \Xi^i_{c_i,\, c_{i+1}}$, let $\mathcal{O}$ be its Z_i-orbit, and let $\mathcal{O}_j \subset \mathbb{C}^{2\,n_j}$ be the $Z_{J_{\lambda_j}}$-orbit of $x[j] = ([z_{j,1}, \cdots, z_{j,n_j}], [y_{j,1}, \cdots, y_{j,n_j}])$ (where the coordinates for x are as in (4.1)). It follows directly from our above remarks that

$$\mathcal{O} \simeq \mathcal{O}_1 \times \cdots \times \mathcal{O}_r, \tag{4.19}$$

where the isomorphism is Z_i-equivariant. Using this description of a Z_i-orbit $\mathcal{O} \subset \Xi^i_{c_i,\, c_{i+1}}$, it is easy to describe the structure of the isotropy groups for the Z_i-action.

Lemma 4.14. *Let $x \in \Xi^i_{c_i,c_{i+1}}$ and let $Stab(x) \subset Z_i$ be the isotropy group of x under the action of Z_i on $\Xi^i_{c_i,c_{i+1}}$. Then, up to reordering,*

$$Stab(x) = \prod_{j=1}^{q} Z_{J_{\lambda_j}} \times \prod_{j=q+1}^{r} U_j, \tag{4.20}$$

where $U_j \subset Z_{J_{\lambda_j}}$ is a unipotent Zariski closed subgroup (possibly trivial) for some q, $0 \leq q \leq r$.

Proof. Suppose that $x \in \Xi^i_{c_i,\, c_{i+1}}$ is given by (4.1). For ease of notation, we let $Z_{J_{\lambda_k}} = Z_{J_k}$. By equation (4.19), to compute the stabilizer of x we need only compute the stabilizers for each of the Z_{J_k} orbits $\mathcal{O}_k = Z_{J_k} \cdot x[k]$, where $1 \leq k \leq r$. To compute the stabilizer of $x[k]$, suppose that $y_{k,i} \neq 0$, but for $i < l \leq n_k$, $y_{k,l} = 0$. We consider the matrix equation:

$$A_k \cdot \underline{y_k} = \underline{y_k}, \tag{4.21}$$

where $A_k \in Z_{J_k}$ is an invertible upper triangular Toeplitz matrix and $\underline{y_k} \in \mathbb{C}^{n_k}$ is the column vector $\underline{y_k} = (y_{k,1}, \cdots, y_{k,i}, 0, \cdots, 0)^T$. As A_k is an upper triangular Toeplitz matrix, we see by considering the ith row in equation (4.21) that A_k is forced to be unipotent. If on the other hand, all $y_{k,j} = 0$ for $1 \leq j \leq n_k$, we can argue similarly using the $z_{k,j}$ and the dual action.

If $y_{k,l} = 0$ for all l and $z_{k,l} = 0$ for all l, then clearly the stabilizer of $x[k]$ is Z_{J_k} itself. Repeating this analysis for each k, $1 \leq k \leq r$ and after possibly reordering the Jordan blocks of x_i, we get the desired result.

Q.E.D.

We have an immediate corollary to the lemma which we stated before Theorem 4.9 as Lemma 4.8.

Corollary 4.15. For any $x \in \Xi^i_{c_i,c_{i+1}}$ $Stab(x)$ is connected.

Proof. Upon reordering the eigevalues, we can always assume that $Stab(x)$ has the form given in (4.20) in Lemma 4.14. This proves the result, since unipotent algebraic groups are always connected and the groups $Z_{J_{\lambda_j}}$ are connected, since they are centralizers of regular elements in $\mathfrak{gl}(n_j)$.

Q.E.D.

We can now prove the structural theorem about the group Z_i that lets us construct the morphism $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ in the general case.

Theorem 4.16. *Let $x \in \Xi^i_{c_i,\,c_{i+1}}$ and let $Stab(x) \subset Z_i$ denote the isotropy group of x under the action of Z_i on $\Xi^i_{c_i,\,c_{i+1}}$. Then as an algebraic group,*

$$Z_i = Stab(x) \times K,$$

where K is a connected, Zariski closed algebraic subgroup of Z_i.

Proof. For the purposes of this proof we denote by H the group $Stab(x)$. Without loss of generality, we assume H is as given in (4.20). Let $\mathfrak{z}_i = Lie(Z_i)$ and let $\mathfrak{h} = Lie(H)$. Now, by Lemma 4.14, $\mathfrak{h}$

$$\mathfrak{h} = \bigoplus_{j=1}^{q} \mathfrak{z}_{J_{\lambda_j}} \oplus \bigoplus_{j=q+1}^{r} \mathfrak{n}_j, \tag{4.22}$$

where $\mathfrak{z}_{J_{\lambda_j}}$ is the Lie algebra of the abelian algebraic group $Z_{J_{\lambda_j}}$ and $\mathfrak{n}_j = Lie(U_j)$ is a Lie subalgebra of $\mathfrak{n}^+(n_j)$, the strictly upper triangular matrices in $\mathfrak{gl}(n_j)$.

The proof proceeds in two steps. We first find an algebraic Lie subalgebra $\mathfrak{k} \subset \mathfrak{z}_i$ such that $\mathfrak{z}_i = \mathfrak{h} \oplus \mathfrak{k}$ as Lie algebras. We then show that if $K \subset Z_i$ is the corresponding Zariski closed subgroup $Z_i = H\,K$ and $H \cap K = \{e\}$. To find $\mathfrak{k}$, consider the abelian Lie algebra $\mathfrak{z}_{J_{\lambda_j}}$ for $q+1 \leq j \leq r$. Since $\mathfrak{z}_{J_{\lambda_j}}$ is abelian, it has a Jordan decomposition as a direct sum of Lie algebras $\mathfrak{z}_{J_{\lambda_j}} = \mathfrak{z}^{ss}_{J_{\lambda_j}} \oplus \mathfrak{z}^n_{J_{\lambda_j}}$, where $\mathfrak{z}^{ss}_{J_{\lambda_j}}$ are the semisimple elements of $\mathfrak{z}_{J_{\lambda_j}}$ and $\mathfrak{z}^n_{J_{\lambda_j}}$ are the nilpotent elements. Now the Lie algebra $\mathfrak{n}_j$ in (4.22) is a subalgebra of $\mathfrak{z}^n_{J_{\lambda_j}}$. Take $\widetilde{\mathfrak{n}_j}$ so that $\mathfrak{z}^n_{J_{\lambda_j}} = \mathfrak{n}_j \oplus \widetilde{\mathfrak{n}_j}$. Let

$$\mathfrak{m}_j = \mathfrak{z}^{ss}_{J_{\lambda_j}} \oplus \widetilde{\mathfrak{n}_j}.$$

Note that $\mathfrak{m}_j \oplus \mathfrak{n}_j = \mathfrak{z}_{J_{\lambda_j}}$. We claim that $\mathfrak{m}_j$ is an algebraic subalgebra of $\mathfrak{z}_{J_{\lambda_j}}$. Indeed, $\widetilde{\mathfrak{n}_j}$ is algebraic, since it is a nilpotent Lie algebra (see [13], pg 383). Let $\widetilde{N_j}$ be the corresponding algebraic subgroup. Then $M_j = \mathbb{C}^\times \times \widetilde{N_j}$ has $Lie(M_j) = \mathfrak{m}_j$, as $\mathbb{C}^\times$ is the semisimple part

of group $Z_{J_{\lambda_j}}$ (see (4.1)). We then take

$$\mathfrak{k} = \bigoplus_{j=q+1}^{r} \mathfrak{m}_j.$$

This finishes the first step.

Let $K = \prod_{j=q+1}^{r} M_j$ be the Zariski closed, connected algebraic subgroup of $\prod_{j=q+1}^{r} Z_{J_{\lambda_j}}$ that corresponds to the algebraic Lie algebra $\mathfrak{k}$. We now show that $Z_i = H \times K$. By our choice of K, $H \cap K$ is finite. But we also have, $H \cap K \subset \prod_{j=q+1}^{r} U_j$ and is thus unipotent (see (4.20)). Since any unipotent group must be connected, we have $H \cap K = \{e\}$. Now, it is clear that $Z_i = H\,K$, as $H\,K$ is a closed, connected subgroup of Z_i of dimension $\dim Z_i$. This completes the proof.

Q.E.D.

With Theorem 4.16 in hand, we can define the general $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ morphism of (4.2), as we did in the strongly regular case. Now, suppose we are given Z_i-orbits in $\Xi^i_{c_i,c_{i+1}}$, $\mathcal{O}^i_{a_i} = K_{a_i} \cdot x_{a_i} \simeq K_{a_i}$ with K_{a_i} as in Theorem 4.16 for $1 \le i \le n-1$, and with $\mathcal{O}^i_{a_i}$ consisting of regular elements of $\mathfrak{gl}(i+1)$ for $1 \le i \le n-2$. We define a morphism

$$\Gamma_n^{a_1,\cdots,a_{n-1}} : K_{a_1} \times \cdots \times K_{a_{n-1}} \to \mathfrak{gl}(n)_c \cap S,$$

as in equation (4.4).

Propositions 4.3 and 4.7, Theorem 4.6, and Remark 4.10 from the strongly regular case remain valid in this case by simply replacing the groups Z_i by the groups K_{a_i}. We recall that the main ingredient in proving Theorem 4.6 is the fact that the group Z_i acts freely on $\mathcal{O}^i_{a_i}$. The analogue of Theorem 4.9 remains valid in this, as it is easy to show

$$T_y(Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}) = V_y,$$

with V_y as in (2.7).

We obtain at last Theorem 4.2.

Theorem 4.17. *The image of the map $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ is exactly one A-orbit in $\mathfrak{gl}(n)_c \cap S$. Moreover, every A-orbit in $\mathfrak{gl}(n)_c \cap S$ is of the form $Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ for some choice of orbits $\mathcal{O}^i_{a_i} \subset \Xi^i_{c_i,c_{i+1}}$, with $\mathcal{O}^i_{a_i}$ consisting of regular elements of $\mathfrak{gl}(i+1)$ for $1 \le i \le n-2$.*

The following corollary of Thoerem 4.2 is a generalization of Theorem 3.14 in [9] to include elements that are not necessarily strongly regular.

Corollary 4.18. Let $x \in \mathfrak{gl}(n)_c \cap S$. The A-orbit of x, $A \cdot x$ is a smooth, irreducible subvariety of $\mathfrak{gl}(n)$ that is isomorphic as an algebraic variety to a closed subgroup $K_{a_1} \times \cdots \times K_{a_{n-1}}$ of the connected algebraic group $Z_1 \times \cdots \times Z_{n-1}$.

5. Counting A-orbits in $\mathfrak{gl}(n)_c^{sreg}$

Using Theorem 4.9, we can count the number of A-orbits in $\mathfrak{gl}(n)_c^{sreg}$ for any $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$ and explicitly describe the orbits. From Theorem 4.9 and Remark 4.10 counting the number of A-orbits in $\mathfrak{gl}(n)_c^{sreg}$ is equivalent to counting the number of Z_i-orbits in $\Xi^i_{c_i, c_{i+1}}$ on which Z_i acts freely. We show in this section that the number of such orbits is directly related to the number of degeneracies in the roots of the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ (see (1.2)). The study of this problem can be reduced to studying the structure of nilpotent solution varieties $\Xi^i_{0,0}$. Thus, we begin our discussion by describing the A-orbit structure of the nilfibre $\mathfrak{gl}(n)_0^{sreg}$.

5.1. **Nilpotent solution varieties and A-orbits in the nilfibre.** In this section, we study strongly regular matrices in the fibre $\mathfrak{gl}(n)_0$. By definition $x \in \mathfrak{gl}(n)_0$ if and only if $x_i \in \mathfrak{gl}(i)$ is nilpotent for all i. Such matrices have been studied by [11] and [12].

We restate Definition 4.1 of the solution variety $\Xi^i_{c_i, c_{i+1}}$ in this case. Elements of $\mathfrak{gl}(i+1)$ of the form

$$X = \begin{bmatrix} 0 & 1 & \cdots & 0 & y_1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & y_i \\ z_1 & \cdots & \cdots & z_i & w \end{bmatrix} \tag{5.1}$$

which are nilpotent define the nilpotent solution variety at level i, which we denote by $\Xi^i_{0,0}$. In this case, it is easy to write down elements in $\Xi^i_{0,0}$. For example, we can take all of the z_j, y_j, and w to be 0. However, such an element is not regular, and so cannot be used to construct a $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ mapping that gives rise to a strongly regular orbit in $\mathfrak{gl}_0^{sreg}(n)$. To describe A-orbits in $\mathfrak{gl}(n)_0^{sreg}$, we focus our attention on free Z_i-orbits in $\Xi^i_{0,0}$, (see Theorem 4.9). To find such orbits, we need to compute the characteristic polynomial of X.

Proposition 5.1. *The characteristic polynomial of the matrix in (5.1) is*

$$\det(X - t) = (-1)^i \left[-t^{i+1} + wt^i + \textstyle\sum_{l=0}^{i-1} \sum_{j=1}^{i-l} z_j y_{j+l} t^{i-1-l} \right]. \tag{5.2}$$

Proof. We compute the characteristic polynomial for the matrix in (5.1) using the Schur complement formula for the determinant (see [7], pgs 21-22). In the notation of that reference $\alpha = \{1, \cdots, n-1\}$ and $\alpha' = \{n\}$. Let $J = X_i$ denote the principal nilpotent Jordan block. Then the Schur complement formula in [7] gives

$$\det(X - t) = \det(J - t)\,(w - t) - \underline{z}\ adj(J - t)\ \underline{y}, \tag{5.3}$$

where $adj(J - t) \in \mathfrak{gl}(i)$ denotes the classical adjoint matrix, $\underline{z} = [z_1, \cdots, z_i]$ is a row vector, and $\underline{y} = [y_1, \cdots, y_i]^T$ is a column vector. We easily compute that $\det(J - t) =$

$(-1)^i t^i$. It is not difficult to see that

$$adj(J-t) = (-1)^{i-1} \begin{bmatrix} t^{i-1} & t^{i-2} & \cdots & \cdots & t & 1 \\ 0 & t^{i-1} & t^{i-2} & \cdots & \cdots & t \\ \vdots & 0 & t^{i-1} & \ddots & & \vdots \\ & & 0 & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & t^{i-2} \\ 0 & \cdots & & \cdots & 0 & t^{i-1} \end{bmatrix}.$$

Now, we compute that the coefficient of t^{i-1-l} for $0 \leq l \leq i-1$ in the product $\underline{z}\ adj(J-t)\ \underline{y}^T$ is

$$(5.4) \qquad (-1)^{i-1} \sum_{j=1}^{i-l} z_j y_{j+l}.$$

Summing up the terms in (5.4) for $0 \leq l \leq i-1$ and using equation (5.3), we obtain the polynomial in (5.2).

Q.E.D.

For the matrix in (5.1) to be nilpotent, we require that all of the coefficients of the polynomial in (5.2) (excluding the leading coefficient) vanish.

$$(5.5) \qquad \begin{gathered} z_1 y_i = 0 \\ z_1 y_{i-1} + z_2 y_i = 0 \\ \vdots \\ z_1 y_1 + \cdots + z_1 y_i = 0 \end{gathered}$$

We claim that $\Xi^i_{0,0}$ has exactly two free Z_i-orbits. These correspond to choosing either $z_1 \in \mathbb{C}^\times$, $y_i = 0$, or $y_i \in \mathbb{C}^\times$, $z_1 = 0$ in the first equation of (5.5). We claim that any point in $\Xi^i_{0,0}$ with $z_1 \neq 0$ is in

$$(5.6) \qquad \mathcal{O}^i_L = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 \\ z_1 & \cdots & \cdots & z_i & 0 \end{bmatrix},$$

with $z_j \in \mathbb{C}\,, 2 \leq j \leq i$. Any point in $\Xi^i_{0,0}$ with $y_i \in \mathbb{C}^\times$ is in

$$\mathcal{O}^i_U = \begin{bmatrix} 0 & 1 & \cdots & 0 & y_1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & y_i \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}, \tag{5.7}$$

with $y_j \in \mathbb{C}\,, 1 \leq j \leq i-1$. To verify this claim, note that if $z_1 \neq 0$ and $y_i = 0$, then $y_1 = 0$, $y_2 = 0, \cdots, y_{i-1} = 0$ by successive use of equations (5.5). The case $y_i \neq 0$, $z_1 = 0$ is similar. An easy computation in linear algebra, as in the proof of Lemma 4.14 gives that Z_i acts freely on $\mathcal{O}^i_U$ and $\mathcal{O}^i_L$. We think of $\mathcal{O}^i_U$ as the "upper orbit " in $\Xi^i_{0,0}$ and $\mathcal{O}^i_L$ as the "lower orbit". Both orbits consist of regular elements of $\mathfrak{gl}(i+1)$ by Theorem 4.13.

Now, suppose that both $z_1 = 0 = y_i$ in (5.5). It is easy to see that such an element has a non-trivial isotropy group in Z_i containing the one dimensional subgroup of matrices

$$\begin{bmatrix} 1 & 0 & \cdots & c \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & \cdots & 1 \end{bmatrix},$$

with $c \in \mathbb{C}^\times$. It does not belong to a Z_i-orbit of dimension i.

Thus, to analyze $\mathfrak{gl}^{sreg}_0(n)$, we consider only the Z_i-orbits $\mathcal{O}^i_U\,, \mathcal{O}^i_L$. Using the orbits $\mathcal{O}^i_U\,, \mathcal{O}^i_L$, we can construct 2^{n-1} morphisms of the form $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ where $a_i = \mathcal{O}^i_U\,, \mathcal{O}^i_L$ for $1 \leq i \leq n-1$. The following result follows immediately from Theorems 4.9 and 4.12 and Remark 4.10.

Theorem 5.2. *The nilfibre $\mathfrak{gl}(n)^{sreg}_0$ contains 2^{n-1} A-orbits. On $\mathfrak{gl}(n)^{sreg}_0$ the orbits of A are orbits of a free action of the algebraic group $(\mathbb{C}^\times)^{n-1} \times \mathbb{C}^{\binom{n}{2}-n+1}$.*

The nilfibre has much more structure than Theorem 5.2 indicates. We can see this additional structure by considering an example of an A-orbit given as the image of a morphism $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ with $\mathcal{O}^i_{a_i} = \mathcal{O}^i_U$, $\mathcal{O}^i_L$ and its closure. Closure here means either closure in the Zariski topology in $\mathfrak{gl}(n)$ or in the Euclidean topology, since A-orbits are constructible sets these two different types of closure agree (see Theorem 3.7 in [9]). For ease of notation, we will abbreviate from now on $\mathcal{O}^i_L = L$, $\mathcal{O}^i_U = U$.

Example 5.3. Let us take our A-orbit in $\mathfrak{gl}(4)^{sreg}_0$ to be the image of $\Gamma_4^{a_1,a_2,a_3}$ with $a_1 = L, a_2 = L, a_3 = U$. For coordinates, let us take for $\mathcal{O}^1_L$, $z_1 \in \mathbb{C}^\times$, for $\mathcal{O}^2_L$, $z_2 \in \mathbb{C}^\times$, $z_3 \in \mathbb{C}$, and for $\mathcal{O}^3_U$, y_1, $y_2 \in \mathbb{C}$, $y_3 \in \mathbb{C}^\times$. In these coordinates, we compute that $Im\Gamma^{L,L,U}$ is

$$Im\Gamma^{L,L,U} = \begin{bmatrix} 0 & 0 & 0 & \frac{y_3}{z_1 z_2} \\ z_1 & 0 & 0 & \frac{y_2}{z_2} - \frac{y_3 z_3}{z_2^2} \\ z_1\, z_3 & z_2 & 0 & y_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.8}$$

Since for $x \in \mathfrak{gl}(n)_c^{sreg}$, $\overline{A \cdot x}$ is an irreducible variety of dimension $\binom{n}{2}$ by Theorem 3.12 in [9], we compute the closure

$$\overline{Im\Gamma^{L,L,U}} = \begin{bmatrix} 0 & 0 & 0 & a_1 \\ a_2 & 0 & 0 & a_3 \\ a_4 & a_5 & 0 & a_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.9}$$

with $a_i \in \mathbb{C}$ for $1 \leq i \leq 6$. $\overline{Im\Gamma^{L,L,U}}$ is a nilradical of a Borel subalgebra that contains the standard Cartan subalgebra of diagonal matrices in $\mathfrak{gl}(4)$. The easiest way to see this is to note that the strictly lower triangular matrices in $\mathfrak{gl}(4)$ are conjugate to $\overline{Im\Gamma^{L,L,U}}$ by the permutation $\tau = (1432)$.

This example illustrates that the A-orbits in $\mathfrak{gl}(n)_0^{sreg}$ are essentially parameterized by prescribing whether or not the the $i \times i$ cutoff of an element $x \in \mathfrak{gl}(n)_0$ has zeroes in its ith column or zeroes in its ith row. This is because for an $x \in \mathfrak{gl}(n)_0$ to be in the image of a morphism $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ with $a_i = L, U$, the ith row or the ith column of x_i must entirely consist of zeroes for each i by Proposition 4.3.

Contrast this with the following example of a matrix $x \in \mathfrak{gl}(n)_0$ each of whose cutoffs is regular, but that is not strongly regular.

Example 5.4. Consider $x \in \mathfrak{gl}(4)_0$

$$x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_3 \\ y_1 & 0 & 0 & 0 \end{bmatrix}, \tag{5.10}$$

where $x_2 \in \mathbb{C}^\times$, $y_1 \in \mathbb{C}^\times$, and $x_3 \in \mathbb{C}$. Note that both the 4th column and row of this matrix have non-zero entries. Thus, this matrix cannot be in the image of a morphism $\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ with $a_i = L, U$ and is not strongly regular. However, one can easily check that each cutoff of this matrix is regular so that $x \in \mathfrak{gl}(4)_0 \cap S$. Thus, $\mathfrak{gl}(4)_0^{sreg}$ is a proper subset of $\mathfrak{gl}(4)_0 \cap S$. (One can also see that this matrix is not strongly regular directly by observing that $\mathfrak{z}_{\mathfrak{gl}(3)}(x_3) \cap \mathfrak{z}_{\mathfrak{gl}(4)}(x) \neq 0$.)

Example 5.3 demonstrates that although the A-orbits $Im\Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ may be complicated, their closures are relatively simple. In this example, the closure is a nilradical of a Borel subalgebra that contains the standard Cartan subalgebra of diagonal matrices in $\mathfrak{gl}(n)$. This is in fact the case in general.

Theorem 5.5. *Let $x \in \mathfrak{gl}(n)_0^{sreg}$ and let $A \cdot x$ denote that A-orbit of x. Then $\overline{A \cdot x}$ is a nilradical of a Borel subalgebra in $\mathfrak{gl}(n)$ that contains the standard Cartan subalgebra of diagonal matrices. More explicitly, if the A-orbit is given by $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ where $a_i = U$ or L for $1 \leq i \leq n-1$, then $\overline{A \cdot x}$ is the set of matrices of the following form*

$$\mathfrak{n}_{a_1,\cdots,a_{n-1}} := \left\{ x : x_{i+1} = \begin{bmatrix} & b_1 \\ x_i & \vdots \\ & b_i \\ 0 & 0 \end{bmatrix} \right\},$$

with $b_j \in \mathbb{C}$ if $a_i = U$, or if $a_i = L$

$$\mathfrak{n}_{a_1,\cdots,a_{n-1}} := \left\{ x : x_{i+1} = \begin{bmatrix} & x_i & & 0 \\ b_1 & \cdots & b_i & 0 \end{bmatrix} \right\},$$

with $b_j \in \mathbb{C}$.

Proof. Let $x \in \mathfrak{gl}(n)_0^{sreg}$. By Gerstenhaber's Theorem [6], it suffices to show the second statement of the theorem. Then $\overline{A \cdot x}$ is a linear space consisting of nilpotent matrices of dimension $\binom{n}{2}$, which is clearly normalized by the diagonal matrices in $\mathfrak{gl}(n)$.

Suppose that $A \cdot x = Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ with $a_i = U, L$. Then it is easy to see $A \cdot x \subset \mathfrak{n}_{a_1,\cdots,a_{n-1}}$ by the definition of the morphism $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ in section 4.2. By Theorem 3.12 in [9], $A \cdot x$ is an irreducible variety of dimension $\binom{n}{2}$. Thus, $\overline{A \cdot x} \subset \mathfrak{n}_{a_1,\cdots,a_{n-1}}$ is an irreducible, closed subvariety of dimension $\binom{n}{2} = \dim \mathfrak{n}_{a_1,\cdots,a_{n-1}}$, and therefore $\overline{A \cdot x} = \mathfrak{n}_{a_1,\cdots,a_{n-1}}$.

Q.E.D.

Remark 5.6. The strictly lower triangular matrices $\mathfrak{n}^-$ is the closure of the A-orbit $\Gamma_n^{L,\cdots,L}$, and the strictly upper triangular matrices $\mathfrak{n}^+$ is the closure of the A-orbit $\Gamma_n^{U,\cdots,U}$.

By Theorem 5.5, the A-orbits in $\mathfrak{gl}(n)_0^{sreg}$ give rise to 2^{n-1} Borel subalgebras of $\mathfrak{gl}(n)$ that contain the diagonal matrices. Moreover, each of the nilradicals $\mathfrak{n}_{a_1,\cdots,a_{n-1}}$ is conjugate to the strictly lower triangular matrices by a unique permutation in $\mathcal{S}_n$, the symmetric group on n letters. The A-orbits in $\mathfrak{gl}(n)_0^{sreg}$ thus determine 2^{n-1} permutations. We now describe these permutations.

Theorem 5.7. *Let $\mathfrak{n}^-$ denote the strictly lower triangular matrices in $\mathfrak{gl}(n)$ and let $\mathfrak{n}_{a_1,\cdots,a_{n-1}}$ be as in Theorem 5.5. Then $\mathfrak{n}_{a_1,\cdots,a_{n-1}}$ is obtained from $\mathfrak{n}^-$ by conjugating by a permutation $\sigma = \tau_1 \tau_2 \cdots \tau_{n-1}$ where $\tau_i \in \mathcal{S}_{i+1}$ is either the long element $w_{i,0}$ of $\mathcal{S}_{i+1}$ or the identity permutation, id_i. The τ_i are determined by the values of a_i as follows. Let $a_n = L$. Starting with $i = n-1$, we compare a_i, a_{i+1}. If $a_i = a_{i+1}$, then $\tau_i = id_i$, but if $a_i \neq a_{i+1}$, then $\tau_i = w_{0,i}$.*

The same procedure beginning with $a_n = U$ produces a permutation that conjugates the strictly upper triangular matrices $\mathfrak{n}^+$ into $\mathfrak{n}_{a_1,\cdots,a_{n-1}}$.

Before proving Theorem 5.7, let us see it in action in Example 5.3. In that case the nilradical in equation (5.9) is $\mathfrak{n}_{L,L,U}$. Thus, according to Theorem 5.7, $\sigma = (13)(14)(23)$, the product of the long elements for $\mathcal{S}_3$ and $\mathcal{S}_4$. Notice that $\sigma = (1432)$, which is precisely the permutation that we observed conjugates the strictly lower triangular matrices in $\mathfrak{gl}(4)$ into $\mathfrak{n}_{L,L,U}$ in Example 5.3.

We now prove Theorem 5.7. In the proof, we will make use of the following notation. Let $\pi_i : \mathfrak{gl}(n) \to \mathfrak{gl}(i)$ be the projection $\pi_i(x) = x_i$. For any subset $S \subset \mathfrak{gl}(n)$ we will denote by S_i the image $\pi_i(S)$.

Proof. Suppose that $L = a_n = a_{n-1} = \cdots = a_{i+1}$, but $a_i = U$. Conjugating $\mathfrak{n}^-$ by $\tau_i = w_{0,i}$ produces the nilradical $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ with $(\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-)_{i+1} = \mathfrak{n}^+_{i+1}$. Thus, $(\mathfrak{n}_{a_1,\cdots,a_{n-1}})_{i+1}$ and $(\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-)_{i+1}$ now have the same $i+1$ columns. We also note that the components of $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ and $\mathfrak{n}_{a_1,\cdots,a_{n-1}}$ in $\mathfrak{gl}(i+1)^\perp$ also agree, as τ_i permutes the strictly lower triangular entries of the rows below the $i+1$th row of $\mathfrak{n}^-$ amongst themselves. Now, we start the procedure again with $(\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-)_{i+1}$ and $a_i = U$ use induction. We note that conjugating $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ by a permutation in $\mathcal{S}_k$ with $k \le i+1$ leaves the component of $\mathrm{Ad}(\tau_i)\cdot\mathfrak{n}^-$ in $\mathfrak{gl}(i+1)^\perp$ unchanged. This proves the theorem.

Q.E.D.

Remark 5.8. There is a related result in recent work of Parlett and Strang. See Lemma 1 in [12], pg 1736.

5.2. **General solution varieties $\Xi^i_{c_i,c_{i+1}}$ and counting A-orbits in $\mathfrak{gl}(n)^{sreg}_c$.** Now, we use our understanding of the nilpotent case to count A-orbits in the general case. Recall the definition of the solution variety $\Xi^i_{c_i,c_{i+1}}$ in section 4.1. We also recall some notation. Given $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$, we write $c = (c_1,\cdots,c_i,\cdots,c_n)$ with $c_i = (z_1,\cdots,z_i) \in \mathbb{C}^i$ and define a corresponding monic polynomial $p_{c_i}(t)$ with coefficients given by c_i (see (1.2)). Recall also that $J = J_{\lambda_1}\oplus\cdots\oplus J_{\lambda_r}$, $J_{\lambda_k} \in \mathfrak{gl}(n_k)$, denotes the regular Jordan form that is the $i\times i$ cutoff of the matrix in (4.1). We now describe the Z_i-orbit structure of the variety $\Xi^i_{c_i,c_{i+1}}$ for any $c_i \in \mathbb{C}^i$ and $c_{i+1} \in \mathbb{C}^{i+1}$.

As in the nilpotent case, to understand $\Xi^i_{c_i,c_{i+1}}$ we must compute the characteristic polynomial of the matrix in (4.1).

Proposition 5.9. *The characteristic polynomial of the matrix in (4.1) is*

(5.11)
$$(w-t)\prod_{k=1}^{r}(\lambda_k - t)^{n_k} + \sum_{j=1}^{r}\left[(-1)^{n_j}\prod_{k=1,k\neq j}^{r}(\lambda_k - t)^{n_k}\sum_{l=0}^{n_j-1}\sum_{j'=1}^{n_j-l} z_{j,j'}y_{j,j'+l}(t-\lambda_j)^{n_j-1-l}\right].$$

The proof of this proposition reduces to the case where J is a single Jordan block of eigenvalue λ. The case of a single Jordan block follows easily from the nilpotent case in Proposition 5.1 by a simple change of variables.

We need to understand the conditions that $z_{i,j}$, $y_{i,j}$, and w must satisfy so that polynomial in (5.11) is equal to the monic polynomial $p_{c_{i+1}}(t)$. w is easily determined by considering the trace of the matrix in (4.1). The values of the $z_{i,j}$ and the $y_{i,j}$ are directly related to the number of roots in common between the polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. Suppose that the polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have j roots in common, where $1 \leq j \leq r$. Then we claim that $\Xi^i_{c_i, c_{i+1}}$ has precisely 2^j free Z_i-orbits. Consider the Jordan block corresponding to the eigenvalue λ_k. First, suppose that λ_k is a root of $p_{c_{i+1}}(t)$. Then Proposition 5.9 implies

$$z_{k,1} y_{k,n_k} = 0. \tag{5.12}$$

However, if λ_k is not a root of $p_{c_{i+1}}(t)$, then Proposition 5.9 gives

$$z_{k,1} y_{k,n_k} \in \mathbb{C}^\times. \tag{5.13}$$

As in the nilpotent case, (5.12) gives rise to two separate cases.

$$z_{k,1} \in \mathbb{C}^\times,\ y_{k,n_k} = 0 \tag{5.14}$$

and

$$y_{k,n_k} \in \mathbb{C}^\times,\ z_{k,1} = 0. \tag{5.15}$$

In the case of (5.14), we can argue using (5.11) that the coordinates $y_{k,i}$ for $1 \leq i \leq n_k$ can be solved uniquely as regular functions of $z_{k,1} \in \mathbb{C}^\times$, $z_{k,2}, \cdots, z_{k,n_k} \in \mathbb{C}$. And in the case of (5.15), we can solve for $z_{k,i}$ as regular functions of $y_{k,n_k} \in \mathbb{C}^\times$ and $y_{k,i} \in \mathbb{C}$, $1 \leq i \leq n_k - 1$. In the case of (5.13), we can take either the $z_{k,i}$ as coordinates that determine the $y_{k,i}$ or vice versa. For concreteness, we take $y_{k,i} = p_i(z_{k,1}, \cdots, z_{k,n_k})$ to be regular functions of $z_{k,1} \in \mathbb{C}^\times$, $z_{k,2}, \cdots, z_{k,n_k} \in \mathbb{C}$.

Remark 5.10. The solutions in the cases of (5.12) and (5.13) are obtained by setting the derivatives of the polynomial in (5.11) up to order $n_p - 1$ evaluated at λ_p equal to the corresponding derivatives of the polynomial $p_{c_{i+1}}(t)$ evaluated at λ_p for $1 \leq p \leq r$. This produces r systems of linear equations. Each system involves only the coordinates $z_{p,k}$ and $y_{p,k}$ from the pth Jordan block. This follows directly from the fact that the eigenvalues λ_s are all distinct. Each system can then be solved inductively using the fact that the coefficient of $(-1)^{n_p}(t-\lambda_p)^q \prod_{k=1, k\neq p}^r (\lambda_k - t)^{n_r}$ is given by the $n - q\,th$ row of the matrix product

$$\begin{bmatrix} z_{p,1} & z_{p,2} & \cdots & z_{p,n_p} \\ 0 & z_{p,1} & \ddots & \vdots \\ \vdots & & \ddots & z_{p,2} \\ 0 & \cdots & 0 & z_{p,1} \end{bmatrix} \cdot \begin{bmatrix} y_{p,1} \\ \vdots \\ \vdots \\ y_{p,n_p} \end{bmatrix}. \tag{5.16}$$

Recall that Z_i is the direct product $Z_i = Z_{J_{\lambda_1}} \times \cdots \times Z_{J_{\lambda_r}}$, with $Z_{J_{\lambda_s}}$ the centralizer of J_{λ_s}. The adjoint action of Z_i on $\Xi^i_{c_i, c_{i+1}}$ is a diagonal action where $Z_{J_{\lambda_s}}$ acts only on the columns and rows of an $x \in \Xi^i_{c_i, c_{i+1}}$ containing J_{λ_s}. This observation allowed us to

decompose a Z_i-orbit $\mathcal{O}$ into the product of $Z_{J_{\lambda_k}}$-orbits, $\mathcal{O}_k \subset \mathbb{C}^{2n_k}$ as in equation (4.19), which we restate here for the convenience of the reader.

$$\mathcal{O} \simeq \mathcal{O}_1 \times \cdots \times \mathcal{O}_k,$$

where the isomorphism is Z_i-equivariant and $Z_{J_{\lambda_k}}$ acts on $\mathcal{O}_k$ as in equation (4.18). If λ_k is a root of $p_{c_{i+1}}(t)$, then (5.12) gives rise to two free $Z_{J_{\lambda_k}}$-orbits, an "upper" orbit $\mathcal{O}_U^k$ in the case of (5.15) and a "lower" orbit $\mathcal{O}_L^k$ in the case of (5.14). This is proved similarly to the nilpotent case. If on the other hand, λ_k is not a root of $p_{c_{i+1}}(t)$, and we have (5.13), then the vector

$$([z_{k,1}, \cdots, z_{k,n_k}], [p_1(z_{k,1}, \cdots, z_{k,n_k}), \cdots, p_k(z_{k,1}, \cdots, z_{k,n_k})]^T) \in \mathbb{C}^{2n_k} \tag{5.17}$$

is a free Z_{J_k}-orbit under the action of Z_{J_k} defined in (4.18). Thus, using the orbits $\mathcal{O}_U^k$ and $\mathcal{O}_L^k$ for $1 \le k \le j$, we can construct 2^j free Z_i-orbits in $\Xi^i_{c_i, c_{i+1}}$ by (4.19).

Now, using Theorem 4.13, we can construct $2^{\sum_{i=1}^{n-1} j_i}\, \Gamma_n^{a_1, a_2, \cdots, a_{n-1}}$ morphisms into $\mathfrak{gl}(n)_c^{sreg}$ where j_i is the number of roots in common to the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. The following result follows immediately from Theorem 4.9 and Theorem 4.12 and Remark 4.10.

Theorem 5.11. *Let $c = (c_1, c_2, \cdots, c_i, c_{i+1}, \cdots, c_n) \in \mathbb{C}^{\frac{n(n+1)}{2}}$. Suppose there are $0 \le j_i \le i$ roots in common between the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$. Then the number of A-orbits in $\mathfrak{gl}(n)_c^{sreg}$ is exactly $2^{\sum_{i=1}^{n-1} j_i}$. Further, on $\mathfrak{gl}(n)_c^{sreg}$ the orbits of A are the orbits of a free algebraic action of the commutative, connected algebraic group $Z = Z_1 \times \cdots \times Z_{n-1}$ on $\mathfrak{gl}(n)_c^{sreg}$.*

Remark 5.12. A similar result is obtained in recent work of Bielawski and Pidstrygach [1]. See Remark 1.3 in the introduction.

Theorem 5.11 lets us identify exactly where the action of the group A is transitive on $\mathfrak{gl}(n)_c^{sreg}$. Let Θ_n be the set of $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$ such that the monic polynomials $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ have no roots in common. From Remark 2.16 in [9], it follows that $\Theta_n \subset \mathbb{C}^{\frac{n(n+1)}{2}}$ is Zariski principal open.

Corollary 5.13. The action of A is transitive on $\mathfrak{gl}(n)_c^{sreg}$ if and only if $c \in \Theta_n$.

Remark 5.14. We will see in the next section that for $c \in \Theta_n$, $\mathfrak{gl}(n)_c^{sreg} = \mathfrak{gl}(n)_c$. Thus, for $c \in \Theta_n$ the fibre $\mathfrak{gl}(n)_c$ consists entirely of strongly regular elements.

Corollary 5.13 allows us to enlarge the set of generic matrices $\mathfrak{gl}(n)_\Omega$ studied by Kostant and Wallach.

5.3. **The new set of generic matrices $\mathfrak{gl}(n)_\Theta$.** We can expand the set of matrices $\mathfrak{gl}(n)_\Omega$ studied by Kostant and Wallach by relaxing the condition that each cutoff is regular semisimple. More precisely, let $\sigma(x_i)$ denote the spectrum of $x_i \in \mathfrak{gl}(i)$, where x_i is viewed as an element of $\mathfrak{gl}(i)$. We define a Zariski open subset of elements of $\mathfrak{gl}(n)$ by

$$\mathfrak{gl}(n)_\Theta = \{x \in \mathfrak{gl}(n) |\ \sigma(x_{i-1}) \cap \sigma(x_i) = \emptyset,\ 2 \le i \le n\}.$$

Clearly, $\mathfrak{gl}(n)_\Theta = \bigcup_{c\in\Theta_n} \mathfrak{gl}(n)_c$.

Theorem 5.15. *The elements of $\mathfrak{gl}(n)_\Theta$ are strongly regular and therefore $\mathfrak{gl}(n)_c^{sreg} = \mathfrak{gl}(n)_c$ for $c \in \Theta_n$. Moreover, $\mathfrak{gl}(n)_\Theta$ is the maximal subset of $\mathfrak{gl}(n)$ for which the action of A is transitive on the fibres of Φ.*

Proof. If $p_{c_i}(t)$ and $p_{c_{i+1}}(t)$ are relatively prime polynomials, then we claim $\Xi^i_{c_i,\, c_{i+1}}$ is exactly one free Z_i-orbit. Indeed, in this case we only have the conditions (5.13) for $1 \le k \le r$. Thus, we can apply our observation in (5.17) to see that $\Xi^i_{c_i,\, c_{i+1}}$ is one free Z_i-orbit and hence consists of regular elements of $\mathfrak{gl}(i+1)$ by Theorem 4.13. Given $x \in \mathfrak{gl}(n)_c$ with $c \in \Theta_n$, we claim that $x \in Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ with $a_i = \Xi^i_{c_i,\, c_{i+1}}$ for $1 \le i \le n-1$. Indeed, $x_2 \in \Xi^1_{c_1,c_2}$ and is therefore regular. Thus, by Remark 4.4, there exists a $g_2 \in GL(2)$ such that $(\mathrm{Ad}(g_2)\cdot x)_3 = (\mathrm{Ad}(g_2)\cdot x_3) \in \Xi^2_{c_2,c_3}$. Now, suppose $x_{i+1} \in \mathrm{Ad}(GL(i))\cdot \Xi^i_{c_i,\, c_{i+1}}$. Thus, $x_{i+1} \in \mathfrak{gl}(i+1)$ is regular and Remark 4.4 provides a $g_{i+1} \in GL(i+1)$ such that $(\mathrm{Ad}(g_{i+1})\cdot x)_{i+2} = \mathrm{Ad}(g_{i+1})\cdot x_{i+2} \in \Xi^{i+1}_{c_{i+1},c_{i+2}}$. By induction, $x_{j+1} \in \mathrm{Ad}(GL(j))\cdot \Xi^j_{c_j,c_{j+1}}$ for any j, $1 \le j \le n-1$. Proposition 4.3 implies that $x \in Im\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$. Thus, by Theorem 4.9, $\mathfrak{gl}(n)_\Theta \subset \mathfrak{gl}(n)^{sreg}$. The rest of the Theorem follows from Corollary 5.13.

Q.E.D.

Remark 5.16. For a matrix $x \in \mathfrak{gl}(n)_c$ where $c \in \Theta_n$, its strictly upper triangular part is determined by its strictly lower triangular part. This follows from the definition of the morphisms $\Gamma_n^{a_1,a_2,\cdots,a_{n-1}}$ and the fact that all of the $y_{k,i}$ can be solved uniquely as regular functions of the $z_{k,i}$ for $1 \le i \le n_k$, $1 \le k \le r$.

The fact that elements of $\mathfrak{gl}(n)_\Theta$ are strongly regular gives us the following corollary.

Corollary 5.17. Let $x \in \mathfrak{gl}(n)_\Theta$. Then $x_i \in \mathfrak{gl}(i)$ is regular for all i.

Using Corollary 5.13 and Theorem 5.11, we get a direct generalization of Theorem 3.23 in [9] for the case of Θ_n.

Corollary 5.18. For $c \in \Theta_n \subset \mathbb{C}^{\frac{n(n+1)}{2}}$, $\mathfrak{gl}(n)_c \simeq Z_1 \times \cdots \times Z_{n-1}$ as algebraic varieties.

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